Exponential growth and decay.

(I) Exponential (unlimited) growth.

Let P_0 and k be positive constants.
We've seen that:

The function $P = P_0 e^{kt}$ satisfies the "exponential (or unlimited) growth" initial value problem (IVP)

$$\underbrace{\frac{dP}{dt} = kP,}_{\text{differential equation (DE)}} \qquad \underbrace{P(0) = P_0.}_{\text{initial condition (IC)}}$$

(E_g)

Terminology: if P denotes population, we call P_0 the <u>initial population</u>.

Also, note that the DE $dP/dt = kP$ may be written

$$k = \frac{dP/dt}{P}. \qquad (*)$$

So: k is the growth rate <u>per individual</u>, and is called the <u>per capita</u> growth rate.

Units for k: if P is in individuals and t in years, then by $(*)$, the units for k are

$$\frac{\text{individuals/year}}{\text{individual}} \qquad (\text{or } \text{year}^{-1}).$$

Example 1.

A population P has per capita growth rate 0.3 year^{-1}, and initial size $P(0) = 100{,}000$. Find:

(a) A formula for $P(t)$ (t in years);

(b) The population four years on.

Solution.

(a) We're given that

$$\frac{dP}{dt} = 0.3P, \qquad P(0) = 100{,}000.$$

So by (E_g),

$$P(t) = 100{,}000e^{0.3t}.$$

(b) By part (a),

$$P(4) = 100{,}000e^{0.3(4)} = 332{,}011.69\ldots \text{ people.}$$

(II) Exponential decay.

Reasoning as above, we find:

> The function $R = R_0 e^{-kt}$ satisfies the "exponential decay" IVP
>
> $$\frac{dR}{dt} = -kR, \qquad R(0) = R_0.$$

(E_d)

Here, $k > 0$ is the (per unit) decay rate; $R_0 > 0$ is the initial amount.

Example 2.

Sugar dissolves in water at a rate proportional to the amount remaining. Write an equation for the amount $S(t)$ of sugar present at time t, in terms of the decay rate k and initial amount S_0 (at $t=0$).

Solution.

We have $\frac{dS}{dt} = -kS$ and $S(0) = S_0$, so by (E_d),

$$S(t) = S_0 e^{-kt}.$$

(III) Basic properties of $y = e^x$.

(i) $e^0 = 1$.

(ii) $e^x > 0$ for all x.

(iii) $e^{-x} = 1/e^x$ for all x.

(iv) $e^{r+s} = e^r e^s$ for all r, s.

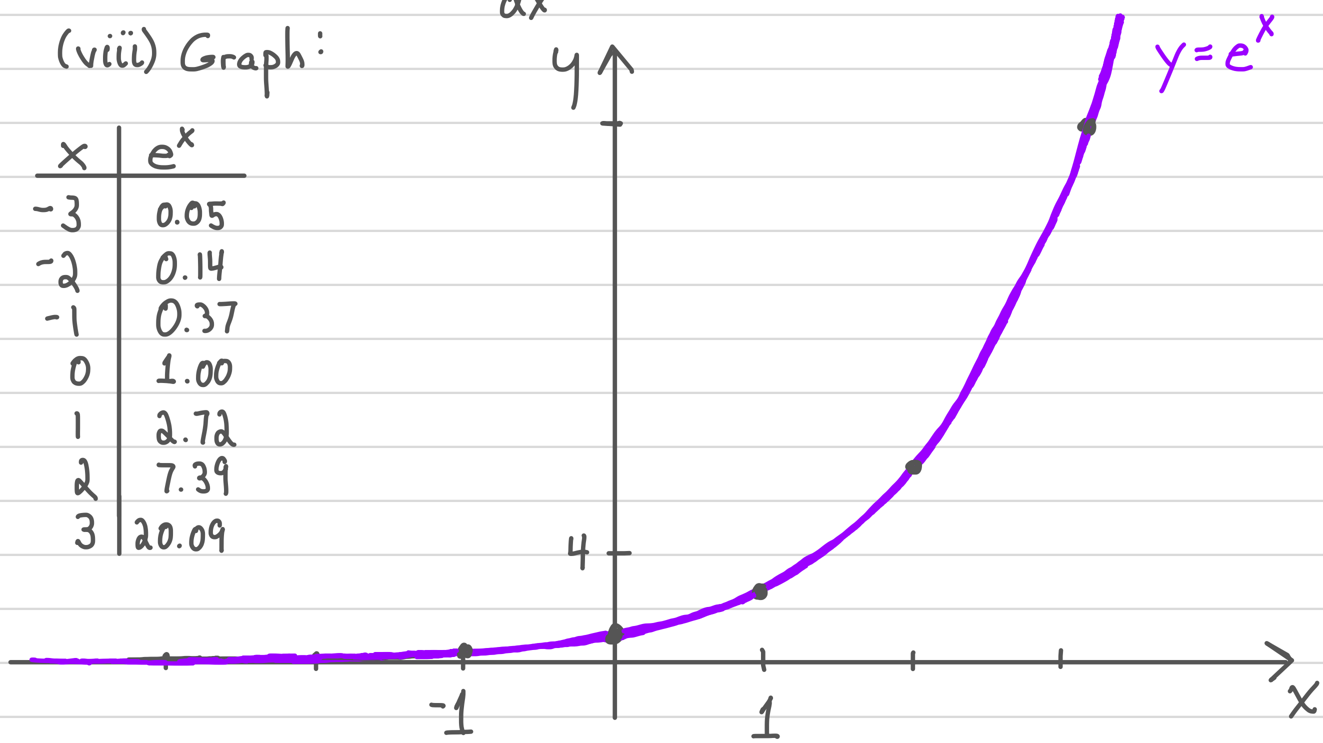
(v) $e^{r-s} = e^r / e^s$ for all r, s

(vi) $(e^r)^s = e^{rs}$ for all r, s.

(vii) $\frac{d}{dx}[e^x] = e^x$.

(viii) Graph:

x	e^x
-3	0.05
-2	0.14
-1	0.37
0	1.00
1	2.72
2	7.39
3	20.09



Example 3. Let $f(x) = e^x e^{7-x}$.

Find $f'(x)$ in two ways:

(a) Differentiate, then simplify;

(b) Simplify, then differentiate.

Solution.

(a) $f'(x) = \frac{d}{dx}\left[e^{x}e^{7-x}\right]$

$= e^{x}\frac{d}{dx}\left[e^{7-x}\right] + e^{7-x}\frac{d}{dx}\left[e^{x}\right]$

$= e^{x}e^{7-x}\frac{d}{dx}\left[7-x\right] + e^{7-x}e^{x}$

$= e^{x}e^{7-x}(-1) + e^{7-x}e^{x} = 0$

(the two summands cancel).

(b) $f(x) = e^{x}e^{7-x} = e^{x+7-x} = e^{7}$ (a <u>constant</u>),

so

$f'(x) = \frac{d}{dx}\left[e^{7}\right] = 0.$