Week 6 – Tuesday, 9/29

## The (natural) exponential function $y = e^x$.

I) Basics.

Recall that

$$\frac{d}{dx}[b^x] = \ln(b)\cdot b^x \qquad (*)$$

for $b > 0$. So, if $e$ is a number with $\ln(e) = 1$, then $(*)$ gives

$$\frac{d}{dx}[e^x] = 1\cdot e^x$$

or, more simply,

$$\boxed{\frac{d}{dx}[e^x] = e^x}$$

$e^x$ is its own derivative!

Note that $e$ is just a number. Which number? Well, recall that

$$\ln(b) = \lim_{\Delta x \to 0} \frac{b^{\Delta x} - 1}{\Delta x}.$$

Plug in $b = e$:

$$1 = \ln(e) = \lim_{\Delta x \to 0} \frac{e^{\Delta x} - 1}{\Delta x}.$$

Solve for $e$:

$$e = \lim_{\Delta x \to 0} (1 + \Delta x)^{1/\Delta x}.$$

We can use this to approximate $e$: we get

$$\boxed{e = 2.7182818284\ldots}$$

Examples. Find:

(1) $\frac{d}{dx}[\sin(e^x)]$

(2) $\frac{d}{dx}[e^{\sin(x)}]$

(3) $\frac{d}{dy}\left[\frac{y}{1+e^{2y}}\right]$.

Solution.

(1) $$\frac{d}{dx}[\sin(e^x)] = \cos(e^x)\cdot\frac{d}{dx}[e^x] = \cos(e^x)\cdot e^x = e^x\cos(e^x).$$

(2) $$\frac{d}{dx}[e^{\sin(x)}] = e^{\sin(x)}\cdot\frac{d}{dx}[\sin(x)] = \cos(x)e^{\sin(x)}.$$

(3) $$\frac{d}{dy}\left[\frac{y}{1+e^{2y}}\right] = \frac{(1+e^{2y})\cdot\frac{d}{dy}[y] - y\frac{d}{dy}[1+e^{2y}]}{(1+e^{2y})^2}$$

$$= \frac{(1+e^{2y})\cdot 1 - y(0+e^{2y}\cdot 2)}{(1+e^{2y})^2} = \frac{1+e^{2y}-2ye^{2y}}{(1+e^{2y})^2}.$$

From a modeling perspective, the big deal about the natural exponential function is this:

Example 4.

Show that, for any constants $P_0$ and $k$, the function

$$P = P_0 e^{kt}$$

satisfies the "initial value problem*"

$$\underbrace{\frac{dP}{dt} = kP}_{\text{a differential equation (DE)}}, \qquad \underbrace{P(0) = P_0}_{\text{an initial condition (IC)}}.$$

* By "initial value problem" we mean: one or more DE's together with one or more IC's.

Solution.

First, the DE:

$$\frac{dP}{dt} = \frac{d}{dt}[P_0 e^{kt}] = P_0 \frac{d}{dt}[e^{kt}]$$

$$= P_0 e^{kt} \cdot \frac{d}{dt}[kt] = \underbrace{P_0 e^{kt}}_{\text{this is } P!} \cdot k$$

$$= P \cdot k = kP,$$ as claimed.

Next, the IC:

$$P(0) = P_0 e^{k \cdot 0} = P_0 \cdot 1 = P_0,$$

also as claimed.