Week 6 - Thursday, 10/1

Logarithms and "how long" problems.

Goal: to solve problems like

$$400{,}000 = 100{,}000 e^{0.3t} \quad \text{or} \quad \frac{1}{2} = e^{-kt}$$

for t.

I) A cool property of natural logarithms (they "undo" natural exponentials).

Recall that

$$\frac{d}{dx}[b^x] = \ln(b) \cdot b^x$$

or, solving for $\ln(b)$:

$$\ln(b) = \frac{\frac{d}{dx}[b^x]}{b^x}.$$

What happens if we plug in $b = e^a$ (where a is any real number)?

We get

$$\ln(e^a) = \frac{\frac{d}{dx}[(e^a)^x]}{(e^a)^x} \overset{\text{properties of } e^x}{=} \frac{\frac{d}{dx}[e^{ax}]}{e^{ax}}$$

$$= \frac{a \cdot \cancel{e^{ax}}}{\cancel{e^{ax}}} = a.$$

"ln gets you out of the exponent"

Conclusion: $\boxed{\ln(e^a) = a}$ for any real number a.

II) Applications to "how long" problems.

Example 1.

Last time, we had a population P given by

$$P(t) = 100{,}000e^{0.3t} \quad (t \text{ in years; } P \text{ in people}).$$

How long does it take for $P(t)$ to reach 400,000?

Solution.

We solve

$$100{,}000e^{0.3t} = 400{,}000$$

for t. First, divide by 100,000:

$$e^{0.3t} = 4.$$

Take "ln" of both sides:

$$\ln(e^{0.3t}) = \ln(4)$$
$$0.3t = \ln(4)$$
$$t = \ln(4)/0.3 = 4.62\ldots \text{ years.}$$

Example 2.

A sample of R grams of radium 226 is known to decay according to the equations

$$\frac{dR}{dt} = -kR, \qquad R(0) = R_0,$$

where R is in grams, t in years, and $k = 0.000428$ year^{-1}:

Find the "half-life" of radium 226, meaning: how long it takes for a sample to reduce by half.

Solution.

By (E_2) of last time, we know that

$$R = R_0 e^{-0.000428t}.$$

We want to know: for which t is $R = \frac{1}{2}R_0$?

So we solve:

$$R_0 e^{-0.000428t} = \frac{1}{2}R_0.$$

Divide by R_0:

$$e^{-0.000428t} = 1/2.$$

Take ln:

$$\ln(e^{-0.000428t}) = \ln(1/2)$$

$$-0.000428t = \ln(1/2)$$

$$t = \frac{-\ln(1/2)}{0.000428} \approx 1{,}619.5 \text{ years.}$$