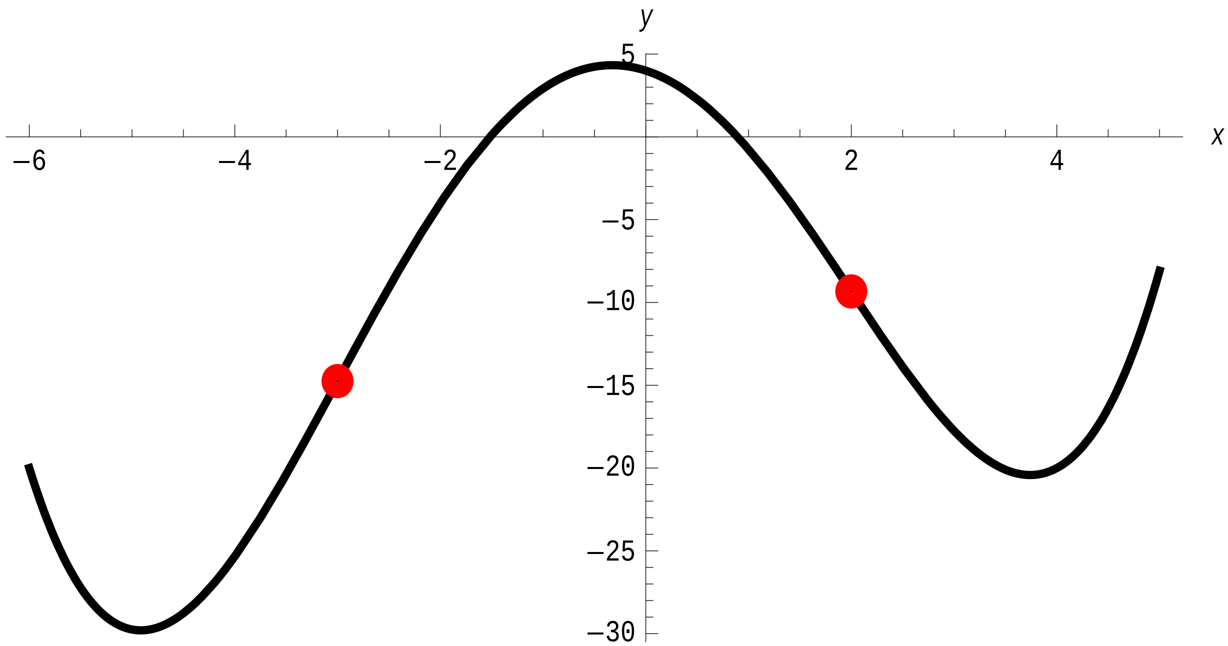
On the axes below is the graph of a certain function $f(x)$.



1. At about which two values of x, between the leftmost and the rightmost low points of $f(x)$ (ignore what's going on beyond these low points), is the graph of $f(x)$ *steepest* (that is, rising or falling most rapidly)? Mark these points of steepest ascent/descent with dots on the above graph.

2. Fill in the blanks below: each blank is to be filled in with one of the following terms (some terms may be used more than once, and some not at all):

   derivative   positive   negative $f(x)$ zero

   To say that the graph of $f(x)$ is at its steepest is to say that the slope, or in other words the ___derivative___, of $f(x)$ is bottoming out (as negative as it can get) or peaking (as ___positive___ as it can get). But we've seen before that, when a function bottoms out or peaks, its ___derivative___ equals zero. SO: to say that the graph of $f(x)$ is at its steepest is to say that the ___derivative___ of the ___derivative___ of $f(x)$ equals ___zero___.

3. Find the derivative $f'(x)$ of the function $f(x)$ graphed above, given that

$$f(x) = \frac{x^4}{12} + \frac{x^3}{6} - 3x^2 - 2x + 4.$$

$$f'(x) = \frac{x^3}{3} + \frac{x^2}{2} - 6x - 2.$$

4. For $f(x)$ as above, find $f''(x)$, which is called the *second derivative* of $f(x)$, and means the derivative of the derivative of $f(x)$. (That is, $f''(x) = \frac{d}{dx}[f'(x)]$.)

$$f''(x) = x^2 + x - 6.$$

5. Solve the equation $f''(x) = 0$ for x. Hint: $f''(x)$ factors nicely.

$$x^2 + x - 6 = 0$$
$$(x - 2)(x + 3) = 0$$
$$x = -3, 2.$$

6. Do your results from exercises 1 and 5 above agree? If not, do you want to adjust one of those answers? Please explain.

   Yes, they agree. The graph is steepest where the second derivative is zero (that is, at $x = -3$ and $x = 2$), as should be the case by the arguments above.

7. Find $f''(x)$ if

$$f(x) = \sin(x^2).$$

At the end of this exercise, in the space provided, indicate which rule(s) (sum, constant multiple, chain, product, quotient) you used. If you used a rule more than once, state how many times you used it. (You don't have to cite any *formulas* you used, just the rules.)

$$f'(x) = \frac{d}{dx}\left[\sin(x^2)\right] = \cos(x^2)\frac{d}{dx}\left[x^2\right] = 2x\cos(x^2).$$

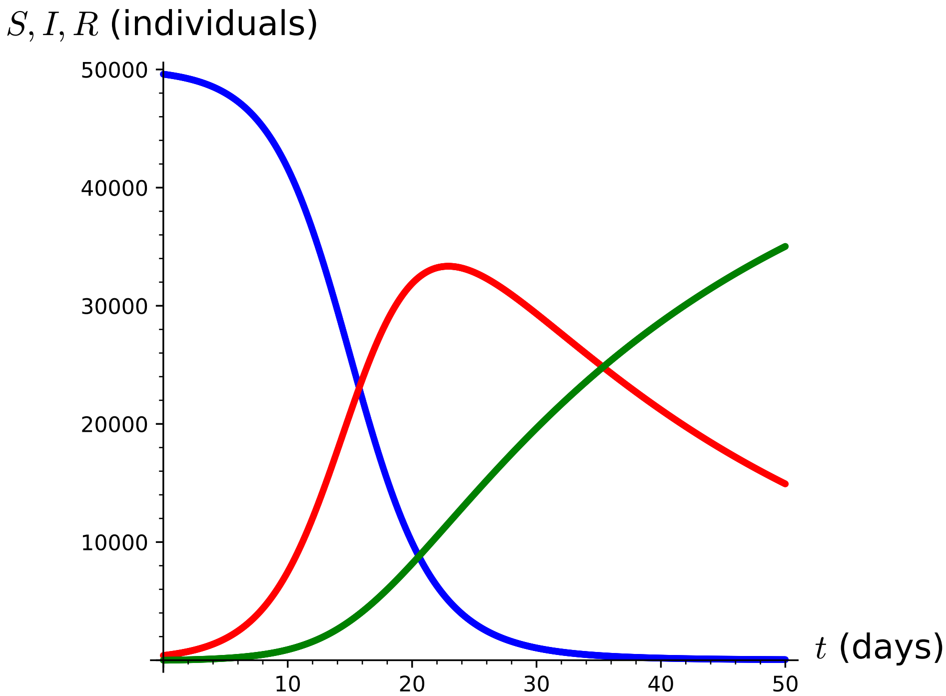
So

$$f''(x) = \frac{d}{dx}\left[2x\cos(x^2)\right] = 2x\frac{d}{dx}\left[\cos(x^2)\right] + \cos(x^2)\frac{d}{dx}\left[2x\right]$$
$$= 2x(-\sin(x^2))\frac{d}{dx}\left[x^2\right] + \cos(x^2)\frac{d}{dx}\left[2x\right] = -4x^2\sin(x^2) + 2\cos(x^2).$$

Rules used: Chain rule (twice); product rule (once); constant multiple rule (once).

8. Consider a disease evolving according to the usual SIR equations

$$
\begin{aligned}
S' &= -a\,S\,I, \\
I' &= \quad a\,S\,I - b\,I, \\
R' &= \qquad\qquad b\,I.
\end{aligned}
$$



Explain carefully why (as the graph suggests) R is at its steepest at the very point where I peaks. Hint: consider differentiating both sides of the equation for R'.

Differentiating the equation $R' = b\,I$ gives $R'' = b\,I'$ (b is a constant; we've used the constant multiple rule). So $R'' = 0$ precisely when $I' = 0$. That is: R is steepest precisely when I peaks.