Week 5 - Wednesday, 2/13

More on the derivative: The Microscope Equation.

Recall: if $f(x)$ is locally linear (differentiable) at $x=a$, then

$$f'(a) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}.$$

So if Δx is <u>small</u>, then

$$\frac{\Delta y}{\Delta x} \approx f'(a)$$

or

$$\boxed{\Delta y \approx f'(a)\Delta x.} \qquad (*)$$

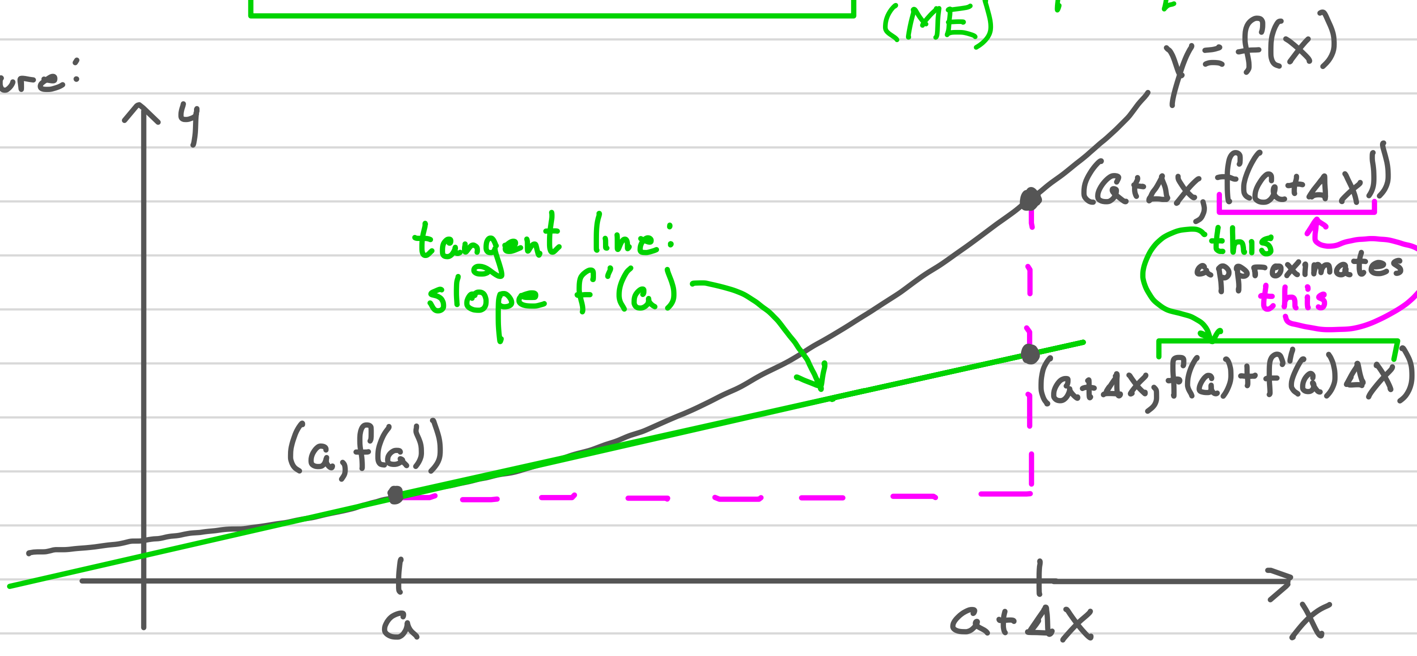
Now $\Delta y = f(a+\Delta x) - f(a)$, so we can rewrite $(*)$:

$$f(a+\Delta x) - f(a) \approx f'(a)\Delta x$$

or

$$\boxed{f(a+\Delta x) \approx f(a) + f'(a)\Delta x.}$$ Microscope Equation (ME)

Picture:



Application to "linear approximation:" if $f(a)$ and $f'(a)$ are easy to find, we can use ME to approximate $f(a+\Delta x)$ at "less easy" nearby points $x = a + \Delta x$!

Example 1.

Approximate $\sqrt{65}$.

Solution.

We know $\sqrt{64}$: we use this, and ME, to approximate $\sqrt{65}$.

We put

$$f(x) = \sqrt{x} \qquad f'(x) = \frac{d}{dx}\left[\sqrt{x}\right] = \frac{d}{dx}\left[x^{1/2}\right] = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}.$$

$$a = 64$$

$$f(a) = \sqrt{64} = 8 \qquad f'(a) = \frac{1}{2\sqrt{64}} = \frac{1}{2\cdot 8} = \frac{1}{16}$$

$$\Delta x = 1$$

So, by ME,

$$\sqrt{65} = f(65) = f(a+\Delta x) \approx f(a) + f'(a)\Delta x$$
$$= 8 + \frac{1}{16}\cdot 1$$
$$= 8\tfrac{1}{16} = 8.0625.$$

[Note: a calculator gives $\sqrt{65} = 8.0622\ldots$]

Example 2.

(a) Write down ME for $g(x) = \frac{1}{(1+x)^2}$ at $x=0$.

(b) Estimate $\frac{1}{0.99^2}$ and $\frac{1}{1.01^2}$.

Solution.

(a) $g(x) = \frac{1}{(1+x)^2}$ $\quad g'(x) = \frac{-2}{(1+x)^3} \cdot \frac{d}{dx}[(1+x)] = \frac{-2}{(1+x)^3}$

$a = 0$

$g(a) = \frac{1}{(1+0)^3} = 1$ $\quad g'(a) = \frac{-2}{(1+0)^3} = -2$

Δx = whatever (arbitrary)

So by ME,

$$g(a+\Delta x) \approx g(a) + g'(a)\Delta x$$
$$g(\Delta x) \approx g(0) + g'(a)\Delta x$$

$$\boxed{\frac{1}{(1+\Delta x)^2} \approx 1 - 2\Delta x} \quad (*)$$

(b) By (*),

$$\frac{1}{0.99^2} = \frac{1}{(1+(-0.01))^2} \approx 1 - 2(-0.01) = 1.02$$

and

$$\frac{1}{1.01^2} = \frac{1}{(1+0.01)^2} \approx 1 - 2(0.01) = 0.98$$

[A calculator gives $\frac{1}{0.99^2} = 1.0203\ldots$; $\frac{1}{1.01^2} = 098029\ldots$.]

Remark: ME is nothing new! Saying

$$f(a+\Delta x) \approx f(a) + f'(a)\Delta x$$

is just saying

$$\text{new } Q \approx \text{old } Q + Q'\Delta x$$

where $Q(x) = f(x)$, a is the "old" point, and $a+\Delta x$ is the "new" one!