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More derivative rules.

We've seen that

$$\frac{d}{dx}[f(x)+g(x)] = f'(x)+g'(x):$$

"the derivative of a sum equals the sum of the derivatives."

Do products behave similarly? Is the derivative of a product equal to the product of the derivatives?

Answer: no! Instead we have

$$\boxed{\frac{d}{dx}[f(x)g(x)] = f(x)g'(x)+g(x)f'(x).}$$

The product rule

In words: to differentiate a product, scale (that is, multiply) each factor by the derivative of the other, then add the results.

Examples. Find

(1) $\frac{d}{dx}[x\cos(x)]$

(2) $\frac{d}{dx}[x^2 2^x]$

(3) $\frac{d}{dx}[x^2 5^{x^3}]$

(4) $\frac{d}{dv}[\tan(v^4 \cdot 4^v)]$

(5) Use the product rule to find the derivative of a quotient; that is, find

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right].$$

Solutions.

(1) $\frac{d}{dx}[x\cos(x)] = x\cdot\frac{d}{dx}[\cos(x)] + \cos(x)\cdot\frac{d}{dx}[x]$

$= x\cdot(-\sin(x)) + \cos(x)\cdot 1 = -x\sin(x)+\cos(x).$

(2) $\frac{d}{dx}[x^2 2^x] = x^2\cdot\frac{d}{dx}[2^x] + 2^x\cdot\frac{d}{dx}[x^2]$

$= x^2\ln(2)2^x + 2^x\cdot 2x = 2^x(x^2\ln(2)+2x).$

(3) $\frac{d}{dx}[x^2 5^{x^3}] = x^2\cdot\frac{d}{dx}[5^{x^3}] + 5^{x^3}\cdot\frac{d}{dx}[x^2]$

$= x^2\ln(5)5^{x^3}\cdot\frac{d}{dx}[x^3] + 5^{x^3}\cdot 2x$

$= x^2\ln(5)\cdot 5^{x^3}\cdot 3x^2 + 5^{x^3}\cdot 2x = 3x^4\ln(5)5^{x^3} + 5^{x^3}\cdot 2x$

$= 5^{x^3}(3x^4\ln(5)+2x).$

(4) $\frac{d}{dv}[\tan(v^4\cdot 4^v)] = \sec^2(v^4\cdot 4^v)\cdot\frac{d}{dv}[v^4\cdot 4^v]$

$= \sec^2(v^4\cdot 4^v)\cdot\left(v^4\cdot\frac{d}{dv}[4^v] + 4^v\cdot\frac{d}{dv}[v^4]\right)$

$= \sec^2(v^4\cdot 4^v)\cdot(v^4\cdot\ln(4)\cdot 4^v + 4^v\cdot 4v^3)$

$= (\ln(4)v^4 4^v + 4v^3 4^v)\sec^2(v^4\cdot 4^v).$

(5) Cheap math trick: turn the quotient into a product (turn something new into something old), then use the product rule.
Like this:

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turn quotient into a product

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{d}{dx}\left[(g(x))^{-1}f(x)\right]$$

product rule

$$= (g(x))^{-1}\frac{d}{dx}[f(x)] + f(x)\frac{d}{dx}\left[(g(x))^{-1}\right]$$

chain rule

$$= (g(x))^{-1}f'(x) + f(x)\cdot\left(-1\cdot(g(x))^{-2}\cdot\frac{d}{dx}[g(x)]\right)$$

rewrite

$$= \frac{f'(x)}{g(x)} - \frac{f(x)g'(x)}{(g(x))^2} = \frac{g(x)f'(x)-f(x)g'(x)}{(g(x))^2}.$$

get a common denominator

Note that we've just <u>proved</u>:

$$\boxed{\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)f'(x)-f(x)g'(x)}{(g(x))^2}}$$

The quotient rule

"lo de hi minus hi de lo,
draw a line and down below
is where you put the square of lo"

Examples.

(1) Find and simplify $\frac{d}{dx}\left[\frac{x^3}{x^2+1}\right]$.

Solution.

$$\frac{d}{dx}\left[\frac{x^3}{x^2+1}\right] = \frac{(x^2+1)\frac{d}{dx}[x^3] - x^3\frac{d}{dx}[x^2+1]}{(x^2+1)^2}$$

$$= \frac{(x^2+1)\cdot 3x^2 - x^3(2x)}{(x^2+1)^2} = \frac{3x^4+3x^2-2x^4}{(x^2+1)^2} = \frac{x^4+3x^2}{(x^2+1)^2}.$$

(2) Find $\frac{d}{dt}\left[\frac{\sin(t^2)}{t^3+5}\right]$.

Solution.

$$\frac{d}{dt}\left[\frac{\sin(t^2)}{t^3+5}\right] = \frac{(t^3+5)\frac{d}{dt}[\sin(t^2)] - \sin(t^2)\frac{d}{dt}[t^3+5]}{(t^3+5)^2}$$

$$= \frac{(t^3+5)\cdot\cos(t^2)\cdot\frac{d}{dt}[t^2] - \sin(t^2)\cdot 3t^2}{(t^3+5)^2}$$

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$$= \frac{2t(t^3+5)\cos(t^2) - 3t^2\sin(t^2)}{(t^3+5)^2}.$$

Food for thought: make sure you understand how to differentiate a:

(a) Sum,
(b) Constant multiple,
(c) Chain,
(d) Product,
(e) Quotient.

five rules!