Week 5 - Friday, 2/15

The "natural" exponential function $y = e^x$.

Recall: for $b > 0$,

$$\frac{d}{dx}[b^x] = \ln(b) \cdot b^x, \qquad (*)$$

where

$$\ln(b) = \lim_{\Delta x \to 0} \frac{b^{\Delta x} - 1}{\Delta x}. \qquad (**)$$

It's known that:

$\ln(2) = 0.69314\ldots$ is less than one;
$\ln(3) = 1.09861\ldots$ is greater than one.

So you'd think there'd be some number b, between 2 and 3, with $\ln(e) = 1$. You'd be right!! Call this number e. So:

Definition.
e is the number with $\ln(e) = 1$.

Facts about e.
(a) $e = 2.71828\ldots$. (See Example 2 below.)
(b) By $(*)$ and the above def'n, we have

$$\frac{d}{dx}[e^x] = \ln(e) \cdot e^x = 1 \cdot e^x$$

or, in short,

$$\boxed{\frac{d}{dx}[e^x] = e^x}$$ Derivative formula (F) (a special case of derivative formula (E))

The function $y = e^x$ equals its own derivative!

<u>Examples.</u> (1) Find:

(A) $\dfrac{d}{dx}[e^{4x}]$ (B) $\dfrac{d}{dz}[e^{\cos(z)}]$ (C) $\dfrac{d}{dz}[\cos(e^{z})]$

(D) $q''(r)$ if $q(r) = e^{r^2}$.

(2) Use (**) to approximate e.

(3) Let P_0 and k be constants. Show that the function
$$y = P_0 e^{kt}$$
satisfies the differential equation
$$\frac{dy}{dt} = ky.$$

<u>Solutions.</u>

(1) (A) $\dfrac{d}{dx}[e^{4x}] = e^{4x} \cdot \dfrac{d}{dx}[4x] = 4e^{4x}$.

(since $\dfrac{d}{dx}[e^x] = e^x$, the <u>chain rule</u> tells us that $\dfrac{d}{dx}[e^{g(x)}] = e^{g(x)} \cdot g'(x)$)

(B) $\dfrac{d}{dz}[e^{\cos(z)}] = e^{\cos(z)} \dfrac{d}{dz}[\cos(z)] = -e^{\cos(z)}\sin(z)$.

(C) $\dfrac{d}{dz}[\cos(e^{z})] = -\sin(e^{z}) \cdot \dfrac{d}{dz}[e^{z}] = -e^{z}\sin(e^{z})$.

(D) $q'(r) = e^{r^2} \cdot \dfrac{d}{dr}[r^2] = 2re^{r^2}$.

So
$$q''(r) = \frac{d}{dr}[2re^{r^2}] = 2r\frac{d}{dr}[e^{r^2}] + e^{r^2}\frac{d}{dr}[2r]$$

$= 2r \cdot 2re^{r^2} + e^{r^2} \cdot 2$

$= 4r^2 e^{r^2} + 2e^{r^2} = (4r^2+2)e^{r^2}.$

(2) (**) says

$$\ln(b) = \lim_{\Delta x \to 0} \frac{b^{\Delta x} - 1}{\Delta x}.$$

So, if Δx is <u>small</u>,

$$\ln(b) \approx \frac{b^{\Delta x} - 1}{\Delta x}.$$

Put in $b = e$, and recall that $\ln(e) = 1$:

$$1 \approx \frac{e^{\Delta x} - 1}{\Delta x}.$$

Do some <u>algebra</u>:

$$\Delta x \approx e^{\Delta x} - 1$$
$$\Delta x + 1 \approx e^{\Delta x}$$
$$(\Delta x + 1)^{1/\Delta x} \approx (e^{\Delta x})^{1/\Delta x} = e.$$

Now pick a small Δx, say $\Delta x = 10^{-9}$. We get

$$e \approx (10^{-9}+1)^{1/10^{-9}} = (1.000000001)^{10^9} = 2.718281\ldots$$

(3) Let $y = P_0 e^{kt}$.

Then

$$\frac{dy}{dt} = \frac{d}{dt}[P_0 e^{kt}]$$
$$= P_0 \frac{d}{dt}[e^{kt}] = P_0 e^{kt} \cdot \frac{d}{dt}[kt]$$
$$= \underbrace{P_0 e^{kt}} \cdot k$$

remember: this <u>is</u> y $= y \cdot k = ky$,

as claimed.