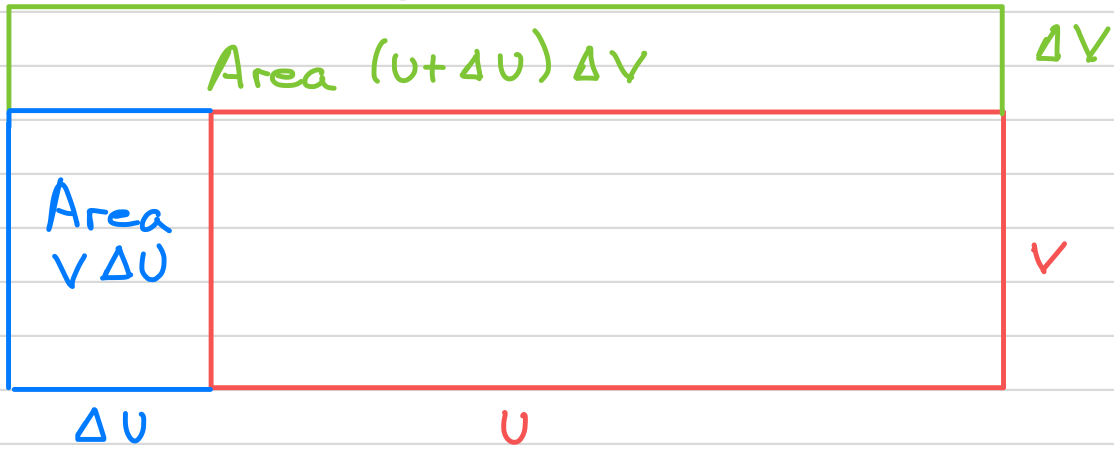
Week 5 - Tuesday, 9/22

The product rule (how to differentiate a product of functions).

Question. Suppose, over Δt units of time, quantities u and v change by Δu and Δv respectively. By how much does the product uv change?

Let's call this change $\Delta(uv)$. Picture:



We see that

$$\Delta(uv) = (u+\Delta u)\Delta v + v\Delta u.$$

Divide by Δt to find average rates of change:

$$\frac{\Delta(uv)}{\Delta t} = (u+\Delta u^{*})\frac{\Delta v}{\Delta t} + v\frac{\Delta u}{\Delta t}.$$

Let $\Delta t \to 0$:

$$\boxed{\frac{d[uv]}{dt} = u\frac{dv}{dt} + v\frac{du}{dt}}$$

The product rule, version 1

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assuming the derivatives on the right exist.

* As $\Delta t \to 0$, $\Delta u \to 0$ too, since du/dt is assumed to exist.

Example 1.

$$\frac{d}{dt}[\overbrace{t^3}^{u}\overbrace{\cos(t)}^{v}] = t^3\frac{d}{dt}[\cos(t)] + \cos(t)\frac{d}{dt}[t^3]$$
$$= t^3(-\sin(t)) + \cos(t)\cdot 3t^2$$
$$= -t^3\sin(t) + 3t^2\cos(t).$$

Remark. Writing $u = f(x)$ and $v = g(x)$, we can rewrite the product rule as follows:

$$\boxed{\frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + g(x)f'(x).}$$

The product rule version 2

Example 2.

$$\frac{d}{dx}[\overbrace{x^2}^{f(x)}\overbrace{2^x}^{g(x)}] = x^2\frac{d}{dx}[2^x] + 2^x\frac{d}{dx}[x^2]$$
$$= x^2\cdot\ln(2)2^x + 2^x\cdot 2x$$
$$= \ln(2)x^2 2^x + x2^{x+1}.$$

Example 3.

$$\frac{d}{dx}[x^2 5^{x^3}] = x^2\frac{d}{dx}[5^{x^3}] + 5^{x^3}\frac{d}{dx}[x^2]$$
$$= x^2\cdot\ln(5)5^{x^3}\frac{d}{dx}[x^3] + 5^{x^3}\cdot 2x$$

$$= x^2 \ln(5) 5^{x^3} \cdot 3x^2 + 2x5^{x^3}$$
$$= 3\ln(5) x^4 5^{x^3} + 2x5^{x^3}.$$

[We used the product rule and the chain rule, in that order.]

Example 4.

$$\frac{d}{dv}[\tan(v^4 4^v)] = \sec^2(v^4 4^v) \frac{d}{dv}[v^4 4^v]$$
$$= \sec^2(v^4 4^v)\left(v^4 \frac{d}{dv}[4^v] + 4^v \frac{d}{dv}[v^4]\right)$$
$$= \sec^2(v^4 4^v)(\ln(4) v^4 4^v + 4v^3 4^v).$$

[We used the chain rule and the product rule, in that order.]

Example 5.

Use the product rule to find
$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right].$$

Solution.

The "cheap trick" is to write the quotient as a product, and then use the product rule. Like this:

turn quotient into a product

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{d}{dx}[(g(x))^{-1} f(x)]$$
$$\overset{\text{product rule}}{=} (g(x))^{-1} \frac{d}{dx}[f(x)] + f(x) \frac{d}{dx}[(g(x))^{-1}]$$

chain rule $\quad = (g(x))^{-1} f'(x) + f(x) \cdot \left(-1 \cdot (g(x))^{-2} \cdot \frac{d}{dx}[g(x)]\right)$

rewrite $\quad = \frac{f'(x)}{g(x)} - \frac{f(x)g'(x)}{(g(x))^2} = \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}$.

get a common denominator

Note that we've just proved:

$$\boxed{\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}}$$

The quotient rule

"lo de hi minus hi de lo,
draw a line and down below
is where you put the square of lo"

Example 6. Find $\frac{d}{dx}\left[\frac{\sin(x^2)}{x^3+5}\right]$.

Solution.
By the quotient rule,

$$\frac{d}{dx}\left[\frac{\sin(x^2)}{x^3+5}\right] = \frac{(x^3+5)\frac{d}{dx}[\sin(x^2)] - \sin(x^2)\frac{d}{dx}[x^3+5]}{(x^3+5)^2}$$

$$= \frac{(x^3+5)\cdot 2x\cos(x^2) - 3x^2\sin(x^2)}{(x^3+5)^2}.$$

[We used the quotient and chain rules, in that order.]