

Week 5 - Thursday, 9/24

More on the derivative: The Microscope Equation.

Recall: if $f(x)$ is locally linear (differentiable) at $x=a$, then

$$f'(a) = \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x}.$$

So if Δx is small, then

$$\frac{\Delta y}{\Delta x} \approx f'(a)$$

or

$$\Delta y \approx f'(a) \Delta x. \quad (*)$$

Now $\Delta y = f(a+\Delta x) - f(a)$, so we can rewrite (*):

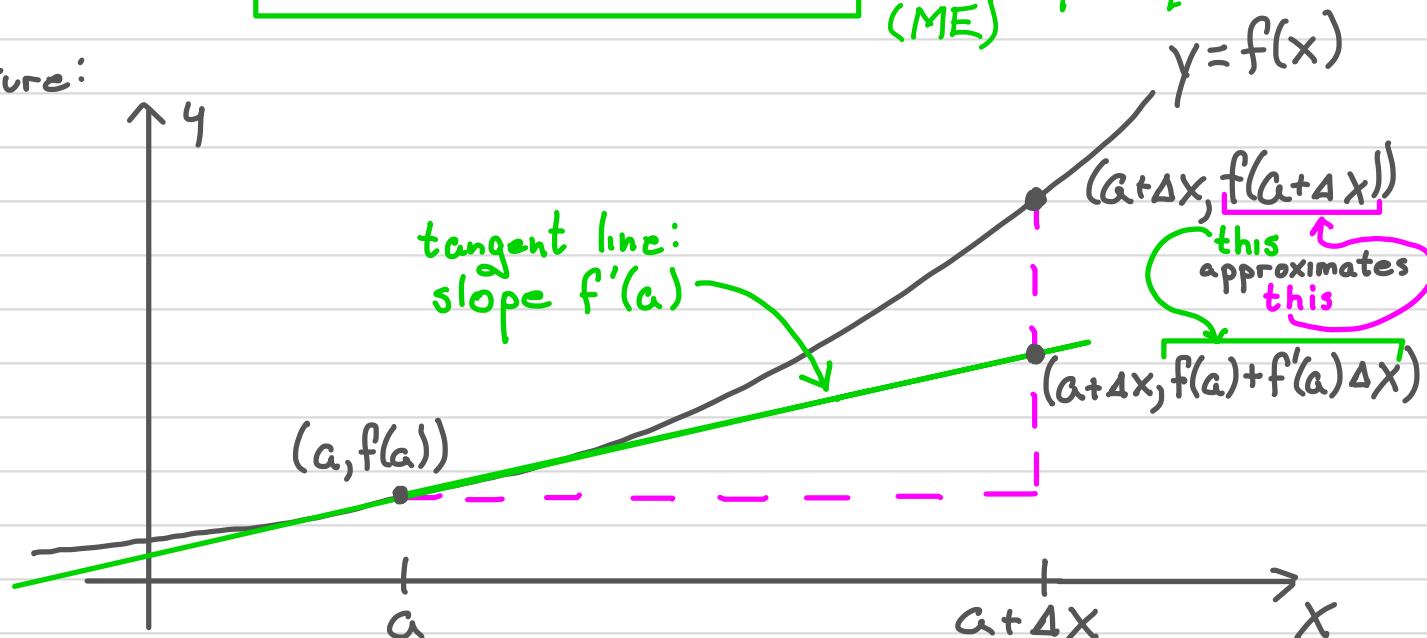
$$f(a+\Delta x) - f(a) \approx f'(a) \Delta x$$

or

$$f(a+\Delta x) \approx f(a) + f'(a) \Delta x.$$

Microscope Equation
(ME)

Picture:



Week 5 - Thursday, 9/24

Application to "linear approximation": if $f(a)$ and $f'(a)$ are easy to find, we can use ME to approximate $f(a+\Delta x)$ at "less easy" nearby points $x = a + \Delta x$.

Example 1.Approximate $\sqrt{65}$.Solution.

We know $\sqrt{64}$: we use this, and ME, to approximate $\sqrt{65}$.

We put

$$f(x) = \sqrt{x} \quad f'(x) = \frac{d}{dx}[\sqrt{x}] = \frac{d}{dx}[x^{1/2}] = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}$$

$$f(a) = \sqrt{64} = 8$$

$$a = 64$$

$$f'(a) = \frac{1}{2\sqrt{64}} = \frac{1}{2 \cdot 8} = \frac{1}{16}$$

$$\Delta x = 1$$

So, by ME,

$$\begin{aligned} \sqrt{65} = f(65) &= f(a + \Delta x) \approx f(a) + f'(a)\Delta x \\ &= 8 + \frac{1}{16} \cdot 1 \\ &= 8 \frac{1}{16} = 8.0625. \end{aligned}$$

[Note: a calculator gives $\sqrt{65} = 8.0622\dots$]

Example 2.

(a) Write down ME for $g(x) = \frac{1}{(1+x)^2}$ at $x=0$.

(b) Estimate $\frac{1}{0.99^2}$ and $\frac{1}{1.01^2}$.

Week 5 - Thursday, 9/24

Solution.

$$(a) \quad g(x) = \frac{1}{(1+x)^2} \quad g'(x) = \frac{-2}{(1+x)^3} \cdot \frac{d}{dx} [(1+x)] = \frac{-2}{(1+x)^3}$$

$$g(a) = \frac{1}{(1+0)^2} = 1 \quad a=0 \quad g'(a) = \frac{-2}{(1+0)^3} = -2$$

 $\Delta x = \text{whatever (arbitrary)}$

So by ME,

$$g(a+\Delta x) \approx g(a) + g'(a)\Delta x$$

$$g(\Delta x) \approx g(0) + g'(0)\Delta x$$

$$\boxed{\frac{1}{(1+\Delta x)^2} \approx 1 - 2\Delta x} \quad (*)$$

(b) By (*),

$$\frac{1}{0.99^2} = \frac{1}{(1+(-0.01))^2} \approx 1 - 2(-0.01) = 1.02$$

and

$$\frac{1}{1.01^2} = \frac{1}{(1+0.01)^2} \approx 1 - 2(0.01) = 0.98$$

[A calculator gives $\frac{1}{0.99^2} = 1.0203\dots$; $\frac{1}{1.01^2} = 0.98029\dots$]

Remark: ME is nothing new! Saying

$$f(a+\Delta x) \approx f(a) + f'(a)\Delta x$$

is just saying

$$\text{new } Q \approx \text{old } Q + Q' \Delta x$$

where $Q(x) = f(x)$, a is the "old" point, and $a+\Delta x$ is the "new" one!