

The chain rule.

Goal: to differentiate chains, like  $f(g(x))$ .

A) Leibniz' notation for derivatives.

Let  $y = f(x)$ . We write  
 $\frac{dy}{dx}$  for  $f'(x)$ .

[ So all of these mean the same thing:  $f'(x)$ ,  $\frac{d}{dx} [f(x)]$ ,  $\frac{dy}{dx}$ . ]

Example 1: if  $y = x^4 - 3x + 2 + 7\cos(x)$ , then  
 $\frac{dy}{dx} = 4x^3 - 3 - 7\sin(x)$ .

Remarks.

(1)  $\frac{dy}{dx}$  is not a fraction, it's a derivative.

(2) INFORMALLY, we imagine that "as  $\Delta x \rightarrow 0$ ,  $\Delta$ 's become  $d$ 's."

So:

$$f'(x) = \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} = \frac{dy}{dx}.$$

B) The chain rule.

Suppose  $y$  depends on  $u$  and  $u$  depends on  $x$ . Then  $y$  depends on  $x$ , and moreover,

$$\frac{\Delta y}{\Delta x} = \frac{\Delta y}{\Delta u} \cdot \frac{\Delta u}{\Delta x} \quad (\text{its just algebra}).$$

Letting  $\Delta x \rightarrow 0$  and using the INFORMAL note above,

Think of the  $du$ 's as cancelling! Week 4 - Monday, 2/4

we get

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

The chain rule version 1.

Example 2.

Find  $\frac{dy}{dx}$  if  $y = \sin(x^2)$ .

Solution.

Write  $y = \sin(u)$  where  $u = x^2$ .

Then

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} = \frac{d}{du} [\sin(u)] \cdot \frac{d}{dx} [x^2] \\ &= \cos(u) \cdot 2x = \cos(x^2) \cdot 2x = 2x \cos(x^2). \end{aligned}$$

just for aesthetics  
↓  
↑ rewrite in terms of original variable

More examples: find

(3)  $\frac{dy}{dx}$  if  $y = 2^{\tan(x)}$

(4)  $\frac{dy}{dx}$  if  $y = \sin(3x)$

(5)  $\frac{dp}{dz}$  if  $p = (3z + 16z^2)^{7/3}$

(6) Calorie consumption at a given instant, in cal/min, if, at that instant, the elliptical trainer reads 0.31 cal/step and 40 steps/min.

(7) The rate at which the area  $A$  of a circular oil slick is growing, in  $\text{mi}^2/\text{hr}$ , when its radius  $r$  equals 5 mi. and is increasing at 0.35 mi/hr.

Solutions.

(3) Write  $y = 2^u$  where  $u = \tan(x)$ . Then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{d}{du} [2^u] \cdot \frac{d}{dx} [\tan(x)] = k_2 \cdot 2^u \cdot \sec^2(x)$$

$$= k_2 \cdot 2^{\tan(x)} \sec^2(x).$$

(4) Write  $y = \sin(u)$  where  $u = 3x$ . Then

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} = \frac{d}{du} [\sin(u)] \cdot \frac{d}{dx} [3x] = \cos(u) \cdot 3 \\ &= 3 \cos(3x). \end{aligned}$$

(5) Write

$p = u^{7/3}$  where  $u = 3z + 16z^2$ . Then

$$\begin{aligned} \frac{dp}{dz} &= \frac{dp}{du} \cdot \frac{du}{dz} \\ &= \frac{7}{3} u^{4/3} \cdot (3 + 32z) \\ &= \frac{7(3z + 16z^2)^{4/3} (3 + 32z)}{3}. \end{aligned}$$

(6) Let  $y = \text{calories}$ ;  $u = \text{steps}$ ;  $x = \text{time in minutes}$ . Then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = 0.31 \frac{\text{cal}}{\text{step}} \cdot 40 \frac{\text{step}}{\text{min}}$$

$$0.31 \cdot 40 \frac{\text{cal}}{\text{min}} = 12.4 \frac{\text{cal}}{\text{min}}.$$

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(7) We have  $A = \pi r^2$  so, by the chain rule,

$$\begin{aligned}\frac{dA}{dt} &= \frac{dA}{dr} \cdot \frac{dr}{dt} \\ &= 2\pi r \frac{dr}{dt}.\end{aligned}$$

So, when  $r = 5$  and  $dr/dt = 0.35$ ,

$$\frac{dA}{dt} = 2\pi(5)(0.35) = 3.5\pi \approx 10.9956 \text{ mi}^2/\text{hr}.$$