Week 4 - Friday, 2/8

The chain rule, continued.

Recall: if y depends on u and u depends on x, then y depends on x, and

$$\boxed{\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}}$$ The chain rule v. 1.

Let's write this differently: put $y = f(u)$ and $u = g(x)$, so $y = f(g(x))$. Then

$$\frac{dy}{dx} = \frac{d}{dx}[f(g(x))], \quad \frac{dy}{du} = f'(u) = f'(g(x)), \quad \frac{du}{dx} = g'(x).$$

So the chain rule v. 1 now reads

$$\boxed{\frac{d}{dx}[f(g(x))] = f'(g(x))g'(x).}$$ The chain rule v. 2

IN OTHER WORDS, to differentiate a chain:

(1) Differentiate the outer function, and plug the inner function into this derivative; then

(2) Multiply your result by the derivative of the inner function.

[DON'T FORGET STEP 2 !!]

Examples.

(1) $$\frac{d}{dx}[\underbrace{(8x^5+2x^4+3)^{23}}] = 23(8x^5+2x^4+3)^{22} \cdot \frac{d}{dx}[8x^5+2x^4+3]$$

$$= 23(8x^5+2x^4+3)^{22}(40x^4+8x).$$

the outer function is u^{23}, and $\frac{d}{du}[u^{23}] = 23u^{22}$

(2) Find y' if $y = \tan(2^x)$.

Solution.

$$y' = \frac{d}{dx}[\underbrace{\tan(2^x)}] = \sec^2(2^x)\cdot\frac{d}{dx}[2^x]$$

outer function is tan(u), derivative is sec²(u)

$$= \ln(2)2^x\sec^2(2^x).$$

(3) $$\frac{d}{dx}[\cos(\cos(x))] = -\sin(\cos(x))\cdot\frac{d}{dx}[\cos(x)]$$

$$= -\sin(\cos(x))(-\sin(x))$$

$$= \sin(\cos(x))\sin(x).$$

(4) $$\frac{d}{dz}\left[3^{\sin(z)}\right] = \ln(3)3^{\sin(z)}\cdot\frac{d}{dz}[\sin(z)]$$

$$= \ln(3)3^{\sin(z)}\cos(z).$$

(5) $$\frac{d}{dz}\left[3^{\sin(\cos(z))}\right] = \ln(3)3^{\sin(\cos(z))}\cdot\frac{d}{dz}[\sin(\cos(z))]$$

chain rule v. 2

$$= \ln(3)3^{\sin(\cos(z))}\cdot\cos(\cos(z))\cdot\frac{d}{dz}[\cos(z)]$$

chain rule v. 2 again

$$= -\ln(3)\cos(\cos(z))\sin(z)\,3^{\sin(\cos(z))}.$$

[For a chain of <u>three</u> functions, use chain rule <u>twice</u>.]

(6) Find $K'(0)$ if

$K(t) = G(t^5 + 2t - \cos(t))$ and $G'(-1) = 3$.

Solution.

First, find $K'(t)$:

$$K'(t) = \frac{d}{dt}\left[G(t^5+2t-\cos(t))\right] = G'(t^5+2t-\cos(t))\,\frac{d}{dt}\left[t^5+2t-\cos(t)\right]$$
$$= (5t^4+2+\sin(t))\,G'(t^5+2t-\cos(t)).$$

Now plug in $t=0$:

$$K'(0) = (5\cdot 0^4+2+\sin(0))\,G'(0^5+2\cdot 0-\cos(0))$$
$$= 2G'(-1) = 2\cdot 3 = 6.$$

(7) (The chain rule in reverse.)
Find a function $g(z)$ such that
$$g'(z) = \frac{\cos(z)}{(5+4\sin(z))^{10}}.$$

Solution.

We know that "to get a -10^{th} power, we differentiate a -9^{th} power." So let's try
$$g(z) = \frac{1}{(5+4\sin(z))^9}.$$

We check:
$$g'(z) = \frac{-9}{(5+4\sin(z))^{10}}\cdot\frac{d}{dz}\left[5+4\sin(z)\right] = \frac{-36\cos(z)}{(5+4\sin(z))^{10}}.$$

We're off by a factor of -36, so we try
$$g(z) = \frac{-1}{36(5+4\sin(z))^9}.$$

Then
$$g'(z) = \frac{(-9)(-1)}{36(5+4\sin(z))^{10}}\cdot\frac{d}{dz}\left[5+4\sin(z)\right] = \frac{36\cos(z)}{36(5+4\sin(z))^{10}}$$
$$= \frac{\cos(z)}{(5+4\sin(z))^{10}},$$

so the correct guess is $g(z) = \dfrac{-1}{36(5+4\sin(z))^9}$.