Week 4 - Tuesday, 9/15

# Differentiation Formulas and Rules.

(I) Formulas (how to differentiate specific functions).
Using the definition

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x},$$

we can derive these formulas:

| | $f(x)$ | $f'(x)$ |
|---|---|---|
| $(A_0)$ | $1$ | $0$ |
| $(A_1)$ | $x$ | $1$ |
| $(A_2)$ | $x^2$ | $2x$ |
| $(A)$ | $x^p$ ($p$ constant) | $px^{p-1}$ |
| $(B)$ | $\sin(x)$ | $\cos(x)$ |
| $(C)$ | $\cos(x)$ | $-\sin(x)$ |
| $(D)$ | $\tan(x)$ | $\sec^2(x)$ |

definitions: $\tan(x) = \sin(x)/\cos(x)$,
$\sec(x) = 1/\cos(x)$,
$\sec^2(x) = (\sec(x))^2$.

Now what about exponential functions $f(x) = b^x$ $(b>0)$, e.g.

$$f(x) = 2^x,\ 7.43^x,\ (1/3)^x, \text{ etc?}$$

We compute: if $f(x) = b^x$, then

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{b^{x+\Delta x} - b^x}{\Delta x}$$

properties of exponents

$$= \lim_{\Delta x \to 0} \frac{b^x b^{\Delta x} - b^x}{\Delta x} = \lim_{\Delta x \to 0} \frac{b^x(b^{\Delta x}-1)}{\Delta x}$$

$b^x$ is independent of $\Delta x$, so pull the $b^x$ factor out front

$$= b^x \boxed{\lim_{\Delta x \to 0} \frac{b^{\Delta x}-1}{\Delta x}}$$

call this number $\ln(b)$: it depends only on $b$!

pronounced "ell enn of b."

To summarize:

(E) $\boxed{\text{If } f(x) = b^x, \text{ then } f'(x) = \ln(b)\cdot b^x \text{ where } \ln(b) = \lim_{\Delta x \to 0} \frac{b^{\Delta x} - 1}{\Delta x} \text{ is \underline{constant} (with respect to } x).}$

<u>Notes</u>:

(1) So the rate of change of an exponential function is <u>proportional to</u> the function itself!

(2) For any <u>particular</u> $b$, we can approximate $\ln(b)$ by

$$\ln(b) \approx \frac{b^{\Delta x} - 1}{\Delta x} \quad \text{for } \Delta x \text{ small.}$$

E.g.

$$\ln(2) \approx \frac{2^{10^{-5}} - 1}{10^{-5}} = 0.6931, \quad \ln(1/3) \approx \frac{\left(\frac{1}{3}\right)^{10^{-5}} - 1}{10^{-5}} = -1.0986,$$

etc.

II) Rules (for finding more <u>complex</u> derivatives out of simpler ones).

Let's write $\frac{d}{dx}[f(x)]$ for $f'(x)$.

E.g. $\frac{d}{dx}[x^2] = 2x, \quad \frac{d}{dx}[2^x] = \ln(2)\cdot 2^x$, etc.

Then we have:

(A) The "constant multiple" rule:

$$\frac{d}{dx}[cf(x)] = cf'(x) \text{ for any constant } c.$$

(B) The "sum" rule:

$$\frac{d}{dx}[f(x)+g(x)] = f'(x)+g'(x).$$

II) Examples. (DIY: what formulas/rules are we using?)

(i) $\frac{d}{dx}[x^{25}] = 25x^{24}.$

(ii) $\frac{d}{dx}[3x^{25}] = 3\cdot\frac{d}{dx}[x^{25}] = 3\cdot 25x^{24} = 75x^{24}.$

(iii) $\frac{d}{dx}[x^{25}+10] = \frac{d}{dx}[x^{25}] + \frac{d}{dx}[10] = 25x^{24}+0 = 25x^{24}.$

(iv) $\frac{d}{dx}\left[\frac{1}{x^{25}} + \sqrt[25]{x}\right] = \frac{d}{dx}[x^{-25}+x^{1/25}] = -25x^{-26} + \frac{1}{25}x^{-24/25}.$

(v) $\frac{d}{dx}[25^x] = k_{25}\cdot 25^x.$

($k_{25} \approx 3.2188$)

(vi) $\frac{d}{dx}\left[\frac{x^{10}}{47} - \sin(x) + 3\tan(x)\right] = \frac{10x^9}{47} - \cos(x) + 3\sec^2(x).$

(vii) $\frac{d}{dx}[\sin(x^2)] = ?$ Answer Thursday.