Week 4 - Thursday, 9/17

The chain rule.

Goal: to differentiate chains, like $f(g(x))$ (for example, $\sin(x^2)$).

A) Liebniz' notation for derivatives.

Let $y = f(x)$. We write

$$\frac{dy}{dx} \text{ for } f'(x).$$

[So all of these mean the same thing: $f'(x)$, $\frac{d}{dx}[f(x)]$, $\frac{dy}{dx}$.]

Example 1: if $y = x^4 - 3x + 2 + 7\cos(x)$, then

$$\frac{dy}{dx} = 4x^3 - 3 - 7\sin(x).$$

Remarks.

(1) $\frac{dy}{dx}$ is not a fraction, it's a derivative.

(2) INFORMALLY, we imagine that "as $\Delta x \to 0$, Δ's become d's."

So:

$$f'(x) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \frac{dy}{dx}.$$

B) The chain rule.

Suppose y depends on u and u depends on x. Then y depends on x, and moreover,

$$\frac{\Delta y}{\Delta x} = \frac{\Delta y}{\Delta u} \cdot \frac{\Delta u}{\Delta x} \quad \text{(its just algebra).}$$

Letting $\Delta x \to 0$ and using the INFORMAL note above, Think of the du's as cancelling! Week 4 - Thursday, 9/17

we get

$$\boxed{\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx}}$$

The chain rule version 1.

Example 2.

Find $\frac{dy}{dx}$ if $y = \sin(x^2)$.

Solution.

Write $y = \sin(u)$ where $u = x^2$.

Then

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx} = \frac{d}{du}[\sin(u)]\cdot\frac{d}{dx}[x^2]$$

$$= \cos(u)\cdot 2x = \cos(x^2)\cdot 2x = 2x\cos(x^2).$$

just for aesthetics

rewrite in terms of original variable

More examples: find

(3) $\frac{dy}{dx}$ if $y = 2^{\tan(x)}$

(4) $\frac{dy}{dx}$ if $y = \sin(3x)$

(5) $\frac{dp}{dz}$ if $p = (3z + 16z^2)^{7/3}$.

(6) Calorie consumption at a given instant, in cal/min, if, at that instant, the elliptical trainer reads 0.31 cal/step and 40 steps/min.

(7) The rate at which the area A of a circular oil slick is growing, in mi^2/hr, when its radius r equals 5 mi. and is increasing at 0.35 mi/hr.

Solutions.

(3) Write $y = 2^u$ where $u = \tan(x)$. Then

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx} = \frac{d}{du}[2^u]\cdot\frac{d}{dx}[\tan(x)] = \ln(2)\cdot 2^u\cdot \sec^2(x)$$

$$= \ln(2)\cdot 2^{\tan(x)} \sec^2(x).$$

(4) Write $y = \sin(u)$ where $u = 3x$. Then

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx} = \frac{d}{du}[\sin(u)]\cdot\frac{d}{dx}[3x] = \cos(u)\cdot 3$$

$$= 3\cos(3x).$$

(5) Write

$p = u^{7/3}$ where $u = 3z + 16z^2$. Then

$$\frac{dp}{dz} = \frac{dy}{du}\cdot\frac{du}{dz}$$

$$= \frac{7}{3}u^{4/3}\cdot(3+32z)$$

$$= \frac{7(3z+16z^2)^{4/3}(3+32z)}{3}.$$

(6) Let y = calories; u = steps; x = time in minutes. Then

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx} = 0.31\,\frac{\text{cal}}{\cancel{\text{step}}}\cdot 40\,\frac{\cancel{\text{step}}}{\text{min}}$$

$$0.31\cdot 40\,\frac{\text{cal}}{\text{min}} = 12.4\,\frac{\text{cal}}{\text{min}}.$$

(7) We have $A = \pi r^2$ so, by the chain rule,

$$\frac{dA}{dt} = \frac{dA}{dr} \cdot \frac{dr}{dt}$$
$$= 2\pi r \frac{dr}{dt}.$$

So, when $r = 5$ and $dr/dt = 0.35$,

$$\frac{dA}{dt} = 2\pi(5)(0.35) = 3.5\pi \approx 10.9956 \text{ mi}^2/\text{hr}.$$