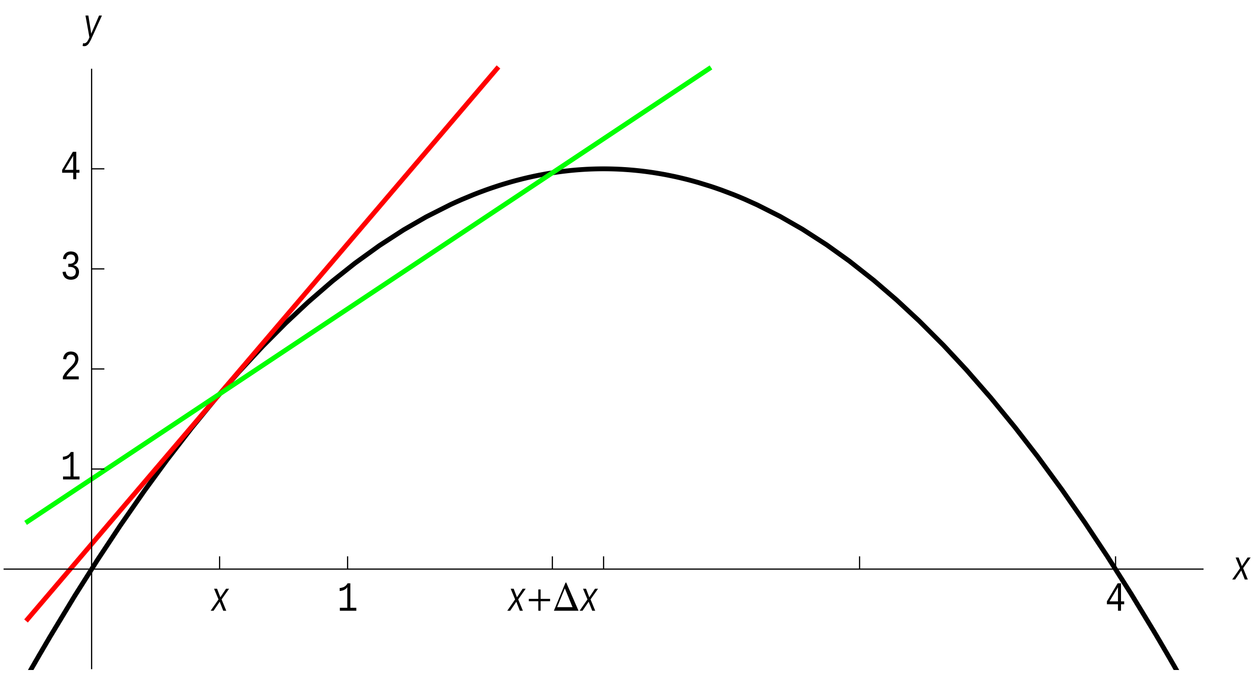
On the axes below is the graph of the function $f(x) = 4x - x^2$.



1. Where (for which values of x) is the graph of $f(x)$ increasing? Where is it decreasing?
   Increasing for $x < 2$; decreasing for $x > 2$.

2. Carefully draw, on the above graph,
   (a) the secant line to the graph of $f(x)$, through the points $(x, f(x))$ and $(x + \Delta x, f(x + \Delta x))$; See green line above.
   (b) the tangent line to the graph of $f(x)$, at the point $(x, f(x))$. See red line above.

3. Fill in the blanks: as Δx approaches ___0___, the above secant line becomes the ___tangent___ line, and the slope $\Delta y/\Delta x$ of this secant line therefore becomes the slope $f'(x)$ of the ___tangent___ line. We call this slope the ___derivative___ of the function $f(x)$ at the point x.

   In other words,

$$f'(x) = \lim_{x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(\underline{\quad x + \Delta x \quad}) - f(x)}{\Delta x}.$$

   In still other words: as ___Δx___ $\to 0$, the average rate of ___change___ $\Delta y/\Delta x$ becomes the ___instantaneous___ rate of change $f'(x)$.

4. Let's do some computations, OK? Heck yes!!.

   Do the *algebra* (oh no, algebra?) (Yes, algebra!) required to complete the following calculation of average rate of change, for the above function $f(x)$. The answer you get at the end *should* be $4 - 2x - \Delta x$.

$$\begin{aligned}\text{average rate of change} = \frac{\Delta y}{\Delta x} &= \frac{f(x+\Delta x) - f(x)}{\Delta x} \\ &= \frac{4(x+\Delta x) - (x+\Delta x)^2 - (4x - x^2)}{\Delta x} \\ &= \frac{4x + 4\Delta x - (x^2 + 2x\Delta x + (\Delta x)^2) - 4x + x^2}{\Delta x} \\ &= \frac{4x + 4\Delta x - x^2 - 2x\Delta x - (\Delta x)^2 - 4x + x^2}{\Delta x} \\ &= \frac{4\Delta x - 2x\Delta x - (\Delta x)^2}{\Delta x} \\ &= \frac{\Delta x(4 - 2x - \Delta x)}{\Delta x} = 4 - 2x - \Delta x.\end{aligned}$$

5. Let's do some *easier* computations, OK?

   OK, but I don't really like the easy ones as much.

   Use your answer to exercise 4 above to complete the following (your final answer should be $4 - 2x$):

   instantaneous rate of change $= \lim_{\Delta x \to 0}$ (average rate of change)

$$= \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} (4 - 2x - \Delta x) = 4 - 2x.$$

6. Fill in the blanks: To summarize what you learned above, if $f(x) = 4x - x^2$, then $f'(x) = \underline{\quad 4 - 2x \quad}$.

7. Where (for which values of x) is the function $f'(x)$ you found in exercise 6 above positive? Where is it negative? (Answer using only the formula for $f'(x)$ you found above.)

   $4 - 2x > 0$ means $4 > 2x$, or $2 > x$. So $f'(x) > 0$ when $2 > x$, or $x < 2$. Similarly, $f'(x) < 0$ when $x > 2$.

8. What do exercises 1 and 7 above have to do with each other?

   They both say the same thing: that $f(x)$ is increasing (equivalently, $f'(x) > 0$) when $x < 2$, and that $f(x)$ is decreasing (equivalently, $f'(x) < 0$) when $x > 2$.

9. A car travels in a straight line, and its position, measured in miles s to the east of some starting point, after t minutes, where t is a number between 0 and 4, is given by

$$s(t) = 4t - t^2.$$

(a) What is the car's velocity, in miles per minute?

Velocity is the rate of change of position with respect to time. So the car's velocity is

$$s'(t) = 4 - 2t.$$

(We've just replaced all the x's and y's in our above computations with t's and s's respectively.)

(b) When is the car's velocity positive, and when is it negative? What does it mean, in terms of the particulars of this situation, to say that the velocity is negative?

The velocity is positive when $4 - 2t > 0$, meaning $t < 2$, and negative when $4 - 2t < 0$, meaning $t > 2$. Negative velocity means the car is moving west, not east.

(c) At what point in time is the car furthest from the starting point? Please explain.

It's furthest when $t = 2$. We can see this either by looking at the graph on page 1, and seeing that $f(x)$ is largest when $x = 2$, or by observing that, at its furthest point, the car must be turning around, so at that instant, its velocity is zero. So at that point, $4 - 2t = 0$, so $t = 2$.