Week 4 - Friday, 9/18

The chain rule, continued.

Recall: if y depends on u and u depends on x, then y depends on x, and

$$\boxed{\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx}}$$

The chain rule v. 1.

Let's write this differently: put $y=f(u)$ and $u=g(x)$, so $y=f(g(x))$. Then

$$\frac{dy}{dx} = \frac{d}{dx}[f(g(x))], \quad \frac{dy}{du} = f'(u) = f'(g(x)), \quad \frac{du}{dx} = g'(x).$$

So the chain rule v. 1 now reads

$$\boxed{\frac{d}{dx}[f(g(x))] = f'(g(x))g'(x).}$$

The chain rule v. 2

IN OTHER WORDS, to differentiate a chain:

(1) Differentiate the outer function, and <u>plug the inner function</u> into this derivative; then

(2) Multiply your result by the <u>derivative</u> of the inner function.

[DON'T FORGET STEP 2 !!]

<u>Examples.</u>

(1) $$\frac{d}{dx}\left[\underbrace{(8x^5+2x^4+3)^{23}}\right] = 23(8x^5+2x^4+3)^{22}\cdot\frac{d}{dx}[8x^5+2x^4+3]$$

the outer function is u^{23}, and $\frac{d}{du}[u^{23}] = 23u^{22}$

$$= 23(8x^5+2x^4+3)^{22}(40x^4+8x).$$

(2) Find $\frac{dy}{dx}$ if $y = \tan(2^x)$.

Solution.

$$\frac{dy}{dx} = \frac{d}{dx}[\underbrace{\tan(2^x)}] = \sec^2(2^x) \cdot \frac{d}{dx}[2^x]$$

outer function is $\tan(u)$, derivative is $\sec^2(u)$

$$= \ln(2) 2^x \sec^2(2^x).$$

(3) $$\frac{d}{dx}[\cos(\cos(x))] = -\sin(\cos(x)) \cdot \frac{d}{dx}[\cos(x)]$$

$$= -\sin(\cos(x))(-\sin(x))$$

$$= \sin(\cos(x))\sin(x).$$

(4) $$\frac{d}{dz}\left[3^{\sin(z)}\right] = \ln(3) 3^{\sin(z)} \cdot \frac{d}{dz}[\sin(z)]$$

$$= \ln(3) 3^{\sin(z)} \cos(z).$$

(5) $$\frac{d}{dz}\left[3^{\sin(\cos(z))}\right] = \ln(3) 3^{\sin(\cos(z))} \cdot \frac{d}{dz}[\sin(\cos(z))]$$

chain rule v. 2

$$= \ln(3) 3^{\sin(\cos(z))} \cdot \cos(\cos(z)) \cdot \frac{d}{dz}[\cos(z)]$$

chain rule v. 2 again

$$= -\ln(3)\cos(\cos(z))\sin(z)\, 3^{\sin(\cos(z))}.$$

[For a chain of <u>three</u> functions, use chain rule <u>twice</u>.]

(6) Find $K'(0)$ if

$K(t) = G(t^5 + 2t - \cos(t))$ and $G'(-1) = 3$.

Solution.

First, find $K'(t)$:

$$K'(t) = \frac{d}{dt}\left[G(t^5+2t-\cos(t))\right] = G'(t^5+2t-\cos(t))\,\frac{d}{dt}\left[t^5+2t-\cos(t)\right]$$
$$= (5t^4+2+\sin(t))\,G'(t^5+2t-\cos(t)).$$

Now plug in $t=0$:

$$K'(0) = (5\cdot 0^4+2+\sin(0))\,G'(0^5+2\cdot 0-\cos(0))$$
$$= 2G'(-1) = 2\cdot 3 = 6.$$

(7) (The chain rule in reverse.)

Find a function $g(z)$ such that
$$g'(z) = \cos(z)(5+4\sin(z))^8.$$

Solution.

We know that "to get an 8^{th} power, we differentiate a 9^{th} power." So let's try
$$g(z) = (5+4\sin(z))^9.$$

We check:
$$g'(z) = 9(5+4\sin(z))^8\cdot\frac{d}{dz}\left[5+4\sin(z)\right] = 9(5+4\sin(z))^8\cdot 4\cos(z)$$
$$= 36\cos(z)(5+4\sin(z))^8.$$

We're off by a factor of 36, so we try

$$g(z) = \frac{(5+4\sin(z))^9}{36}.$$

Then
$$g'(z) = \frac{\cancel{9}(5+4\sin(z))^8\cdot\cancel{4}\cos(z)}{\cancel{36}} = \cos(z)(5+4\sin(z))^8,$$

so we've found the desired $g(z)$.