Week 3 - Wednesday, 1/30

Differentiation Formulas and Rules.

(I) Formulas.

Using the definition

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x},$$

we can derive these formulas:

	$f(x)$	$f'(x)$
(A_0)	1	0
(A_1)	x	1
(A_2)	x^2	$2x$
(A)	x^p (p constant)	px^{p-1}
(B)	$\sin(x)$	$\cos(x)$
(C)	$\cos(x)$	$-\sin(x)$
(D)	$\tan(x)$	$\sec^2(x)$

definitions: $\tan(x) = \sin(x)/\cos(x)$,
$\sec(x) = 1/\cos(x)$,
$\sec^2(x) = (\sec(x))^2$.

Now what about exponential functions $f(x) = b^x$ $(b>0)$, e.g.

$$f(x) = 2^x;\quad 7.43^x;\quad (1/3)^x;\ \text{etc?}$$

We compute: if $f(x) = b^x$, then

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{b^{x+\Delta x} - b^x}{\Delta x}$$

properties of exponents

$$= \lim_{\Delta x \to 0} \frac{b^x b^{\Delta x} - b^x}{\Delta x} = \lim_{\Delta x \to 0} \frac{b^x(b^{\Delta x} - 1)}{\Delta x}$$

b^x is independent of Δx, so pull the b^x factor out front

$$= b^x \boxed{\lim_{\Delta x \to 0} \frac{b^{\Delta x} - 1}{\Delta x}}$$

call this number $\ln(b)$: it depends only on b!

pronounced "ell enn of b."

To summarize:

(E) $\boxed{\begin{array}{l}\text{If } f(x) = b^x \text{, then } f'(x) = \ln(b) \cdot b^x \\ \text{where} \\ \quad \ln(b) = \lim\limits_{\Delta x \to 0} \dfrac{b^{\Delta x} - 1}{\Delta x} \text{ is \underline{constant} (with respect to } x).\end{array}}$

<u>Notes</u>:

(1) So the rate of change of an exponential function is <u>proportional to</u> the function itself!

(2) For any <u>particular</u> b, we can approximate $\ln(b)$ by

$$\ln(b) \approx \frac{b^{\Delta x} - 1}{\Delta x} \quad \text{for } \Delta x \text{ small.}$$

E.g.

$$\ln(2) \approx \frac{2^{10^{-5}} - 1}{10^{-5}} = 0.6931, \quad \ln(1/3) \approx \frac{\left(\frac{1}{3}\right)^{10^{-5}} - 1}{10^{-5}} = -1.0986,$$

etc.

II) Rules.

Let's write $\frac{d}{dx}[f(x)]$ for $f'(x)$.

E.g.

$$\frac{d}{dx}[x^2] = 2x, \quad \frac{d}{dx}[2^x] = \ln(2) \cdot 2^x, \text{ etc.}$$

Then we have:

(A) The "constant multiple" rule:

$$\frac{d}{dx}[cf(x)] = cf'(x) \text{ for any constant } c.$$

(B) The "sum" rule:

$$\frac{d}{dx}[f(x) \underset{\smile}{+} g(x)] = f'(x) \underset{\smile}{+} g'(x).$$

II) Examples. (DIY: what formulas/rules are we using?)

(i) $\frac{d}{dx}[x^{25}] = 25x^{24}.$

(ii) $\frac{d}{dx}[3x^{25}] = 3 \cdot \frac{d}{dx}[x^{25}] = 3 \cdot 25x^{24} = 75x^{24}.$

(iii) $\frac{d}{dx}[x^{25} + 10] = \frac{d}{dx}[x^{25}] + \frac{d}{dx}[10] = 25x^{24} + 0 = 25x^{24}.$

(iv) $\frac{d}{dx}\left[\frac{1}{x^{25}} + \sqrt[25]{x}\right] = \frac{d}{dx}[x^{-25} + x^{1/25}] = -25x^{-26} + \frac{1}{25}x^{-24/25}.$

(v) $\frac{d}{dx}[25^x] = k_{25} \cdot 25^x.$

($k_{25} \approx 3.2188$)

(vi) $\frac{d}{dx}\left[\frac{x^{10}}{47} - \sin(x) + 3\tan(x)\right] = \frac{10x^9}{47} - \cos(x) + 3\sec^2(x).$

(vii) $\frac{d}{dx}[\sin(x^2)] = ?$ Answer Monday.