Derivatives, continued.

We can understand derivatives in three equivalent ways:

Natural: $\left.\begin{array}{l}\text{instantaneous rate} \\ \text{of change of } y=f(x), \\ \text{at } x=a\end{array}\right\} = \left\{\begin{array}{l}\text{limit of average rates of} \\ \text{change of } f(x), \text{ over smaller} \\ \text{and smaller intervals near } x=a\end{array}\right.$

Geometric: $\left.\begin{array}{l}\text{slope of tangent line} \\ \text{to graph of } f(x), \\ \text{at } x=a\end{array}\right\} = \left\{\begin{array}{l}\text{limit, as } \Delta x \to 0, \text{ of slopes of} \\ \text{secant lines through } (a, f(a)) \\ \text{and } (a+\Delta x, f(a+\Delta x))\end{array}\right.$

Algebraic: $$f'(a) = \lim_{\Delta x \to 0} \frac{f(a+\Delta x)-f(a)}{\Delta x}$$

Example 1.

The instantaneous rate of change of $f(x)=x^3$, at $x=2$

= the slope of the tangent line to $f(x)=x^3$, at $x=2$

$$= f'(2) = \lim_{\Delta x \to 0} \frac{f(2+\Delta x)-f(2)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{(2+\Delta x)^3-2^3}{\Delta x} = \lim_{\Delta x \to 0} \frac{\cancel{8}+12\Delta x+6(\Delta x)^2+(\Delta x)^3-\cancel{8}}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{12\Delta x+6(\Delta x)^2+(\Delta x)^3}{\Delta x} = \lim_{\Delta x \to 0} \frac{\cancel{\Delta x}(12+6\Delta x+(\Delta x)^2)}{\cancel{\Delta x}}$$

$$= \lim_{\Delta x \to 0} 12+6\Delta x+(\Delta x)^2 = 12.$$

Example 2

The equation of the line tangent to the graph of $y = x^3$, at $x = 2$, is, by the point-slope form,

$$y = m(x - x_0) + y_0 = f'(2)(x-2) + f(2)$$
$$= 12(x-2) + 2^3$$
$$= 12x - 24 + 8 = 12x - 16$$

(since this tangent line passes through $(x_0, y_0) = (2, f(2))$ and has slope $m = f'(2)$).

Definition. If $f'(a)$ exists, we say f is differentiable, or locally linear, at $x = a$.

COOL FACTS about/interpretations of locally linearity.

(1) Geometrically, "$f(x)$ is locally linear at $x = a$" means "near $x = a$, $f(x)$ quite closely follows its tangent line there." (More on this when we study "The Microscope Equation," soon.) In particular: wherever a function $f(x)$ is differentiable, it "looks linear" (i.e. flat) when you zoom in closely enough.

(2) Geometrically, "$f(x)$ is not locally linear at $x = a$" means "the slope $f'(a)$ of the tangent line to $f(x)$ at $x = a$ is undefined." This can happen in several ways:

(a) This tangent line is vertical ($f'(a)$ is infinite).

(b) This tangent line is ambiguous ($f(x)$ has some kind of corner, cusp, or break at $x = a$).

(3) Examples:

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(a) The functions
$y=|x|$, $y=x^{1/3}$, $y=x^{2/3}$, and

$$y=\begin{cases}-1 & \text{if } x\le 0\\ 1 & \text{if } x>0\end{cases}$$

all fail to be differentiable at $x=0$ (but are differentiable at all other numbers x). See Fig. 2.5, text, p. 69.

(b) Polynomials, cosines, sines, and exponential functions are differentiable everywhere.