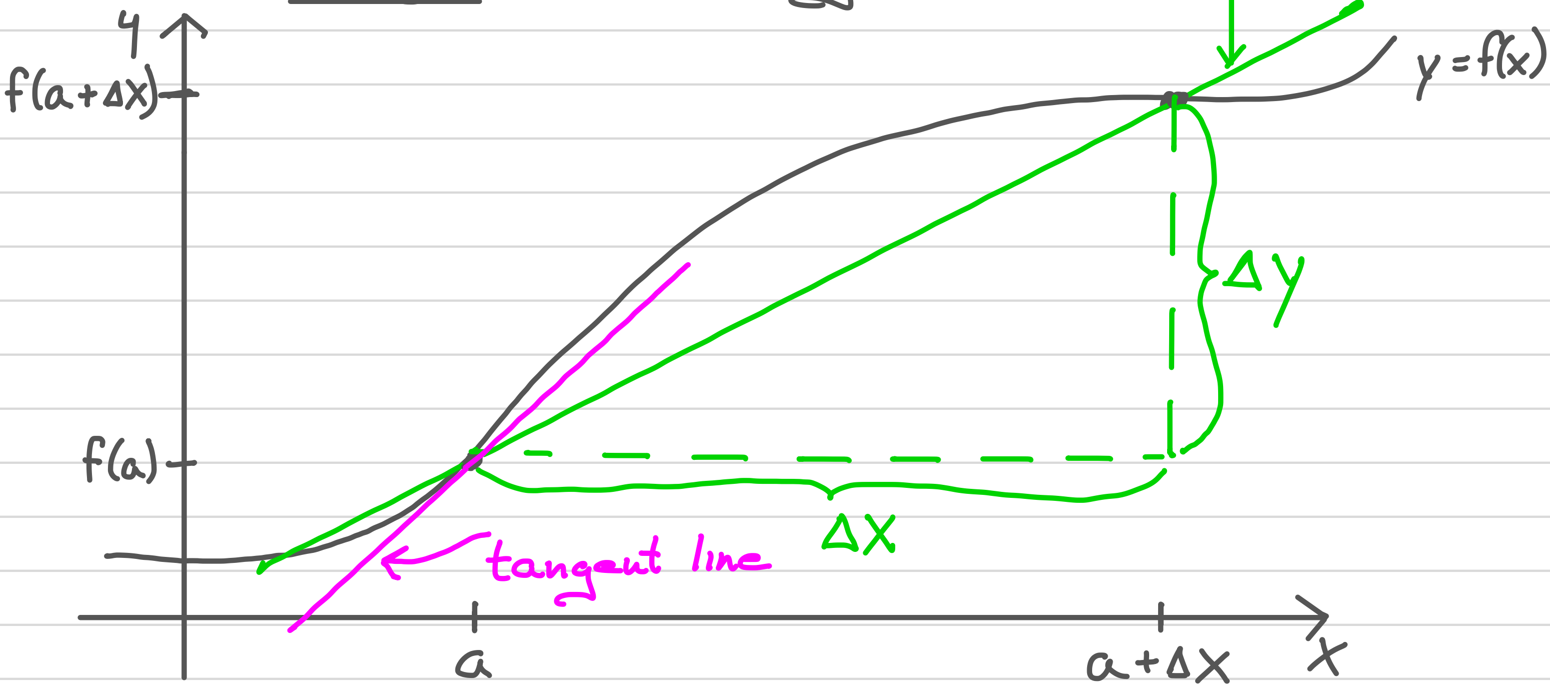
Week 3 - Monday, 1/28

Well yeah, sure, OK, but: just what *is* a rate of change, anyway?

GOAL: given a function $y = f(x)$, and a point $x = a$, to *carefully* define $f'(a)$, the *instantaneous* rate of change of $f(x)$ with respect to x, at $x = a$.

Part I: *average* rates of change.



Suppose x changes from $x = a$ to $x = a + \Delta x$. Then the corresponding change in y is

$$\begin{aligned} \Delta y &= \text{new } y - \text{old } y \\ &= f(a+\Delta x) - f(a). \end{aligned}$$

We define the *average rate of change* of $f(x)$, from $x = a$ to $x = a + \Delta x$, to be the ratio

$$\boxed{\frac{\Delta y}{\Delta x} = \frac{f(a+\Delta x) - f(a)}{\Delta x}}$$

definition of average rate of change

KEY FACT (see picture above):
this average rate of change is the slope of the secant line through the points $(a, f(a))$ and $(a+\Delta x, f(a+\Delta x))$.

Part II: average $\to$ instantaneous.

In the above picture, imagine that we let Δx shrink to zero. Then:

(1) The above secant line becomes the tangent line to the graph of $f(x)$ at $x=a$.

(2) We denote the slope of this tangent line by $f'(a)$.

SO: $f'(a)$ = slope of tangent line
= what happens, as $\Delta x \to 0$, to slopes of secant lines
= what happens, as $\Delta x \to 0$, to $\Delta y / \Delta x$

or, summarizing,

pronounced "the limit, as Δx approaches zero"

$$\boxed{f'(a) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(a+\Delta x) - f(a)}{\Delta x}.}$$

Definition of the derivative $f'(a)$
(= instantaneous rate of change)

Example. Let $f(x) = x^2$. Find:

(a) The average rate of change of $f(x)$, from $x=1$ to $x=1.1$, and from $x=1$ to $x=1.01$.
(b) The average rate of change of $f(x)$ from $x=1$ to $x=1+\Delta x$.
(c) $f'(1)$.
(d) The equation of the tangent line to $f(x)$ at $x=1$.

Solution.

(a) In the first case,
$$\frac{\Delta y}{\Delta x} = \frac{f(1.1)-f(1)}{1.1-1} = \frac{(1.1)^2-1^2}{0.1} = \frac{1.21-1}{0.1} = \frac{0.21}{0.1} = 2.1;$$
in the second,
$$\frac{\Delta y}{\Delta x} = \frac{f(1.01)-f(1)}{1.01-1} = \frac{(1.01)^2-1^2}{0.01} = \frac{1.0201-1}{0.01} = \frac{0.0201}{0.01} = 2.01.$$

(b) $$\frac{\Delta y}{\Delta x} = \frac{f(1+\Delta x)-f(1)}{\Delta x} = \frac{(1+\Delta x)^2-1^2}{\Delta x} = \frac{\cancel{1}+2\Delta x+(\Delta x)^2-\cancel{1}}{\Delta x} = \frac{2\Delta x+\Delta x^2}{\Delta x}$$
$$= \frac{\cancel{\Delta x}(2+\Delta x)}{\cancel{\Delta x}} = 2+\Delta x.$$

by (b) above

(c) $$f'(1) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} (2+\Delta x) = 2$$
since, clearly, $2+\Delta x \to 2$ as $\Delta x \to 0$.

(d) The line in question has slope $m = f'(1) = 2$ and passes through $(1, f(1)) = (1, 1^2) = (1,1)$, so by the point-slope form, it has equation

$$y = 2(x-1)+1 = 2x-2+1 = 2x-1.$$