Goal: To explore some more ideas about modeling with rate equations and SIR.

1. A town of population 100,000 is hit with a measles epidemic, which evolves according to the usual SIR equations

$$\begin{aligned} S' &= -a\,S\,I, \\ I' &= \quad a\,S\,I - b\,I, \\ R' &= \qquad\qquad b\,I. \end{aligned}$$

This unique strain of the measles is known to last for twelve days.

(a) What is the recovery coefficient b, and what are the units for b? Please explain. Express b to six decimal places.

Since the disease lasts twelve days, about 1/12 of those infected recover on any given day, so that $b = 1/12 = 0.083333$. The units of b are day^{-1} (since these are the units that will make the units match up on both sides of the equation $R' = bI$).

On day 15, 14,893 people are susceptible (that is, $S(15)$ =14,893) and 69,613 people are infected (so $I(15)$ =69,613). *One tenth* of a day later, S =13,856.

(b) What, at least approximately, is $S'(15)$ (the rate of change of S at $t = 15$)? What are the units of $S'(15)$? Hint: this rate of change is, at least roughly, the *net change* in S (final value minus initial value) from day 15 to day 15.1, divided by the elapsed time over that period.

By the hint, $S'(15)$ is approximately

$$\frac{S(15.1) - S(15)}{15.1 - 15} = \frac{13856 - 14893}{.1} = -10370$$

individuals per day. (Note that $S'(15)$ is negative, which is indicative of the fact that S is *decreasing*.)

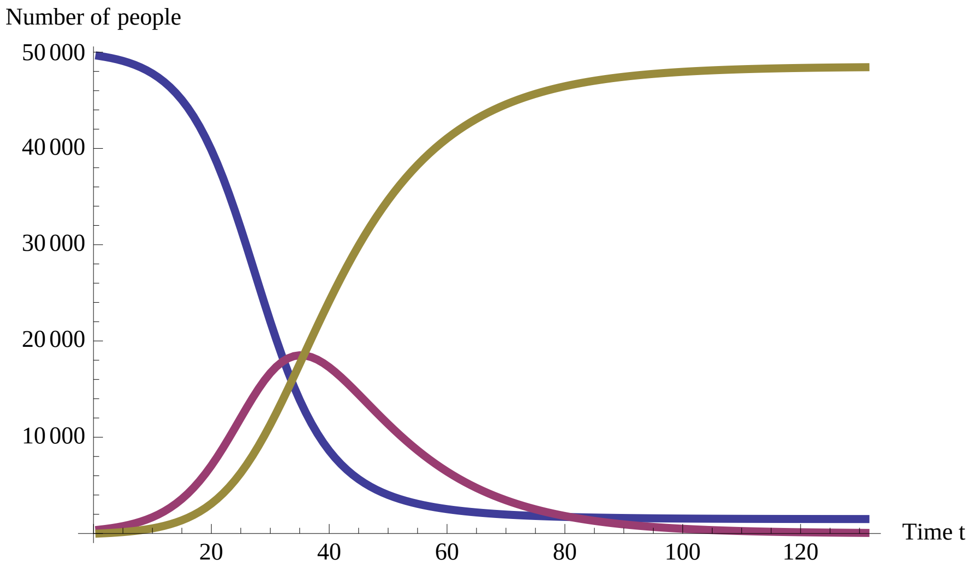
(c) What is the transmission coefficient a? What are the units for a? **HINT:** into above rate equation for S', plug in S', S, and I at $t = 15$, and solve for a. Express a to six decimal places.

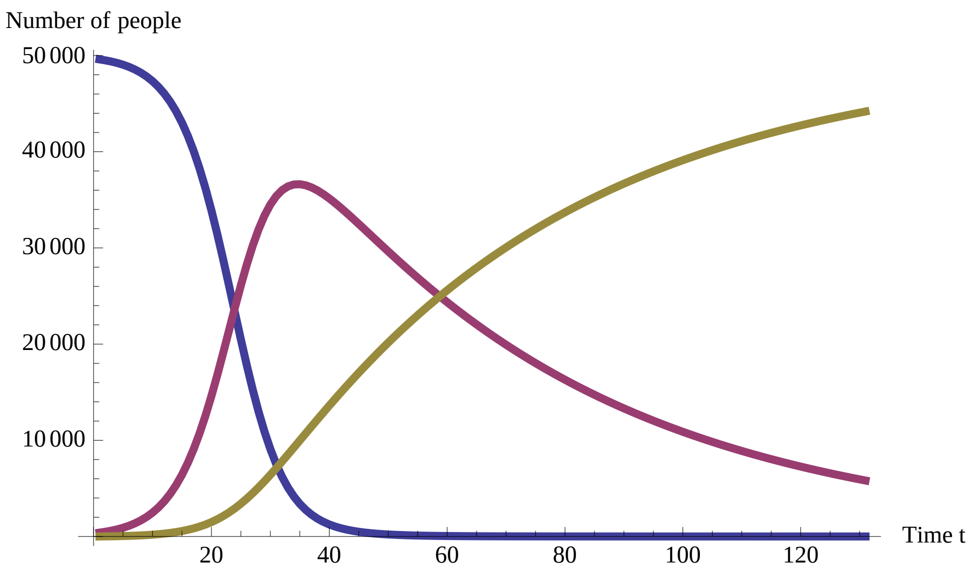
To compute, or at least approximate, a, we can observe the following. We have the equation $S' = -aSI$, so in particular, on day 15,

$$(*) \qquad S'(15) = -a\,S(15)I(15).$$

Plugging $S'(15) = -10370$, $S(15)$ =14893, and $I(15)$ =69613 into $(*)$ gives $-10370 = -a \cdot 14893 \cdot 69613$, or $a = 0.000010$ (to six decimal places). The units of a are (person·day)$^{-1}$ (since these are the units that will make the units match up on both sides of the equation $S' = -aSI$).

2. Pictured below are two graphs depicting evolution of diseases that progress according to the usual SIR model. For both graphs, the initial values $S(0)$, $I(0)$ and $R(0)$, and the transmission coefficient a, are the same. But the two graphs correspond to different recovery coefficients b.





(a) On each of the graphs, label which curve is S, which is I, and which is R.

In each graph, the "backwards S" curve is S, the "bell" curve is I, and the "S" curve is R.

(b) Which of the above two graphs corresponds to the *larger* value of b? Please explain.

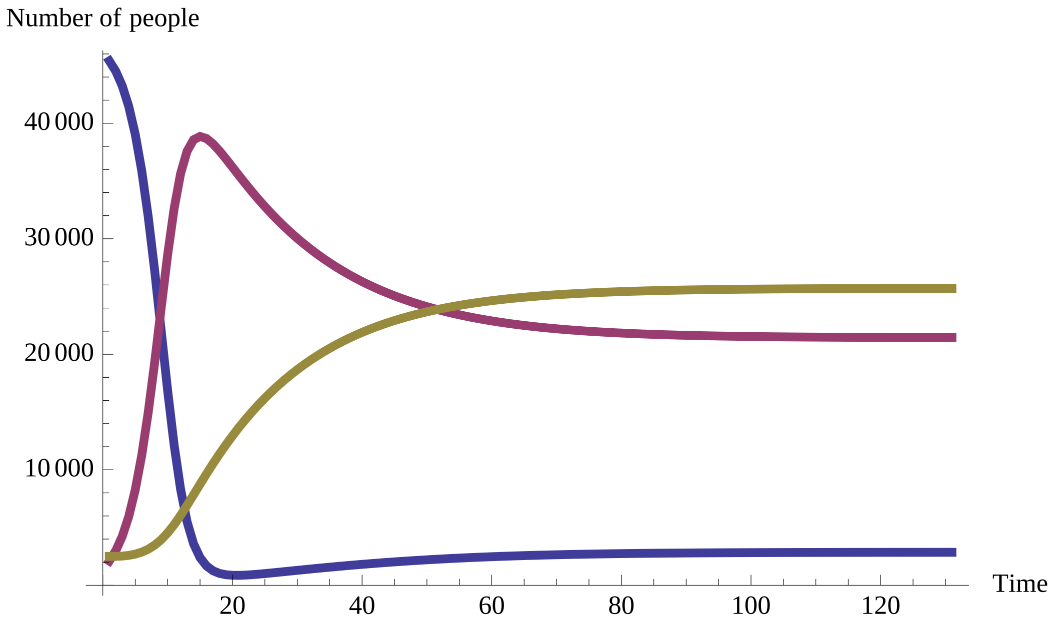
The top one. Remember $b = 1/k$, where k is the number of days to recovery. So larger b means smaller k, which means faster recovery, which we see in the top graph.

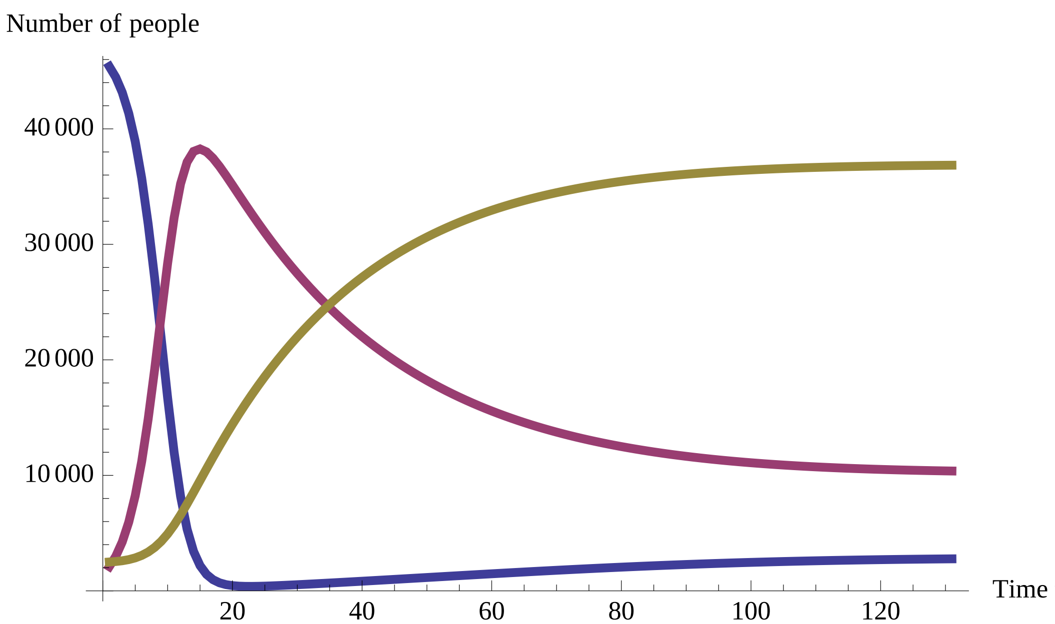
3. Consider an epidemic that progresses according to the usual SIR model, *except* that, now, recovered people become susceptible again (and can infect again) after m days.

(a) *Modify* the usual SIR equations to reflect this new feature (wherein recovered can become susceptible again). HINTS: (a) Your new equations will look a *lot* like the old ones, but with some *new terms* added on. These terms should account for the facts that, now, on average, $1/m$ of the recovered population gets *added to* susceptible population, and *subtracted from* the recovered population, on any given day. (b) Your new equations should involve unspecified parameters a, b, and c, where a and b are as above, and $c = 1/m$.

$$\begin{aligned} S' &= -a\,S\,I \qquad\qquad + cR, \\ I' &= \quad a\,S\,I - b\,I, \\ R' &= \qquad\qquad\quad b\,I - cR. \end{aligned}$$

(b) In the two graphs on the next page, the transmission and recovery coefficients a and b are the same, but the number of days m that it takes to become susceptible again differs from one graph to the next. For which of the two graphs – the one on the top or the one on the bottom — does it take *longer* to become susceptible again? Please explain.





The bottom graph. If it takes longer to become susceptible again, then we would expect the number of infected to level off at a relatively low level, and the number of recovered to level off at a relatively high level, as is happening in the bottom graph.