Derivatives, continued.

We can understand derivatives in three equivalent ways:

Natural: $\left.\begin{array}{l}\text{instantaneous rate}\\\text{of change of } y=f(x),\\\text{at } x=a\end{array}\right\}=\left\{\begin{array}{l}\text{limit of average rates of}\\\text{change of } f(x), \text{ over smaller}\\\text{and smaller intervals near } x=a\end{array}\right.$

Geometric: $\left.\begin{array}{l}\text{slope of tangent line}\\\text{to graph of } f(x),\\\text{at } x=a\end{array}\right\}=\left\{\begin{array}{l}\text{limit, as } \Delta x \to 0, \text{ of slopes of}\\\text{secant lines through } (a, f(a))\\\text{and } (a+\Delta x, f(a+\Delta x))\end{array}\right.$

Algebraic: $$f'(a) = \lim_{\Delta x \to 0} \frac{f(a+\Delta x) - f(a)}{\Delta x}$$

Example 1.

The instantaneous rate of change of $f(x) = x^3$, at $x = 2$

= the slope of the tangent line to $f(x) = x^3$, at $x = 2$

$= f'(2) = \lim_{\Delta x \to 0} \dfrac{f(2+\Delta x) - f(2)}{\Delta x}$

$= \lim_{\Delta x \to 0} \dfrac{(2+\Delta x)^3 - 2^3}{\Delta x} = \lim_{\Delta x \to 0} \dfrac{8 + 12\Delta x + 6(\Delta x)^2 + (\Delta x)^3 - 8}{\Delta x}$

$= \lim_{\Delta x \to 0} \dfrac{12\Delta x + 6(\Delta x)^2 + (\Delta x)^3}{\Delta x} = \lim_{\Delta x \to 0} \dfrac{\Delta x(12 + 6\Delta x + (\Delta x)^2)}{\Delta x}$

$= \lim_{\Delta x \to 0} 12 + 6\Delta x + (\Delta x)^2 = 12.$

Example 2

The equation of the line tangent to the graph of $y = x^3$, at $x = 2$, is, by the point-slope form,

$$y = m(x-x_0) + y_0 = f'(2)(x-2) + f(2)$$
$$= 12(x-2) + 2^3$$
$$= 12x - 24 + 8 = 12x - 16$$

(since this tangent line passes through $(x_0, y_0) = (2, f(2))$ and has slope $m = f'(2)$).

Definition. If $f'(a)$ exists, we say f is <u>differentiable</u>, or <u>locally linear</u>, at $x = a$.

<u>COOL FACTS</u> about/interpretations of locally linearity.

(1) Geometrically, "$f(x)$ is locally linear at $x = a$" means "near $x = a$, $f(x)$ quite closely follows its tangent line there." (More on this when we study "The Microscope Equation," soon.) In particular: wherever a function $f(x)$ is differentiable, it "looks linear" (i.e. flat) when you zoom in closely enough.

(2) Geometrically, "$f(x)$ is <u>not</u> locally linear at $x = a$" means "the slope $f'(a)$ of the tangent line to $f(x)$ at $x = a$ is undefined." This can happen in several ways:

   (a) This tangent line is <u>vertical</u> ($f'(a)$ is <u>infinite</u>).

   (b) This tangent line is <u>ambiguous</u> ($f(x)$ has some kind of corner, cusp, or break at $x = a$).

(3) <u>Examples</u>:

(a) The functions
$$y = |x|, \quad y = x^{1/3}, \quad y = x^{2/3}, \quad \text{and}$$

$$y = \begin{cases} -1 & \text{if } x \leq 0 \\ 1 & \text{if } x > 0 \end{cases}$$

all __fail__ to be differentiable at $x = 0$ (but are differentiable at all other numbers x). See Fig. 2.5, text, p. 69.

(b) Polynomials, cosines, sines, and exponential functions are differentiable everywhere.