Functions and graphs.

A) Basics.

A _function_ is a rule that assigns a _unique_ output to every input.

Call the rule f, the input x, and the corresponding output y; then we can write

function

$$y = f(x).$$

dependent variable   independent variable

Examples.

1) $y = x^2$ (The _implicit_ rule f is given by $f(x) = x^2$.)

2) $y = 3x - 5$: a _linear_ function

3) $y = \dfrac{1}{\sqrt{4-7x}}$

4) $y = x^{1/3} \ (= \sqrt[3]{x}), \quad y = x^{-10} \left(= \dfrac{1}{x^{10}}\right),$

$y = x^{4/7} \left(= \sqrt[7]{x^4} = \left(\sqrt[7]{x}\right)^4\right), \quad y = x^{-4/7} \left(= \dfrac{1}{x^{4/7}}\right),$ etc.

5) $F = \dfrac{9}{5}C + 32$ (Celsius to Fahrenheit)
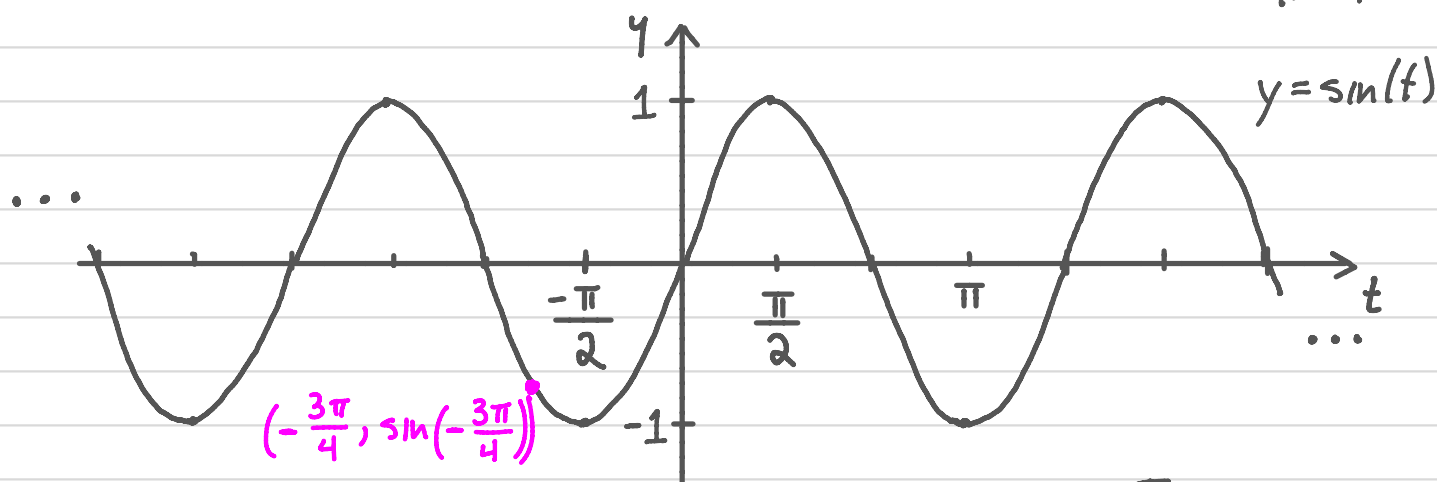
6) $P(t) = \dfrac{100}{1 + 9e^{-t/10}}$

(population, in thousands, after t years, under "logistic growth:" more on this later).

7) $D = q(C)$:

density of water, in kg/m^3, as a function of temperature, in degrees Celsius.

8) $y = \sin(t)$: a trigonometric function, whose graph looks like this:

$y = \sin(t)$

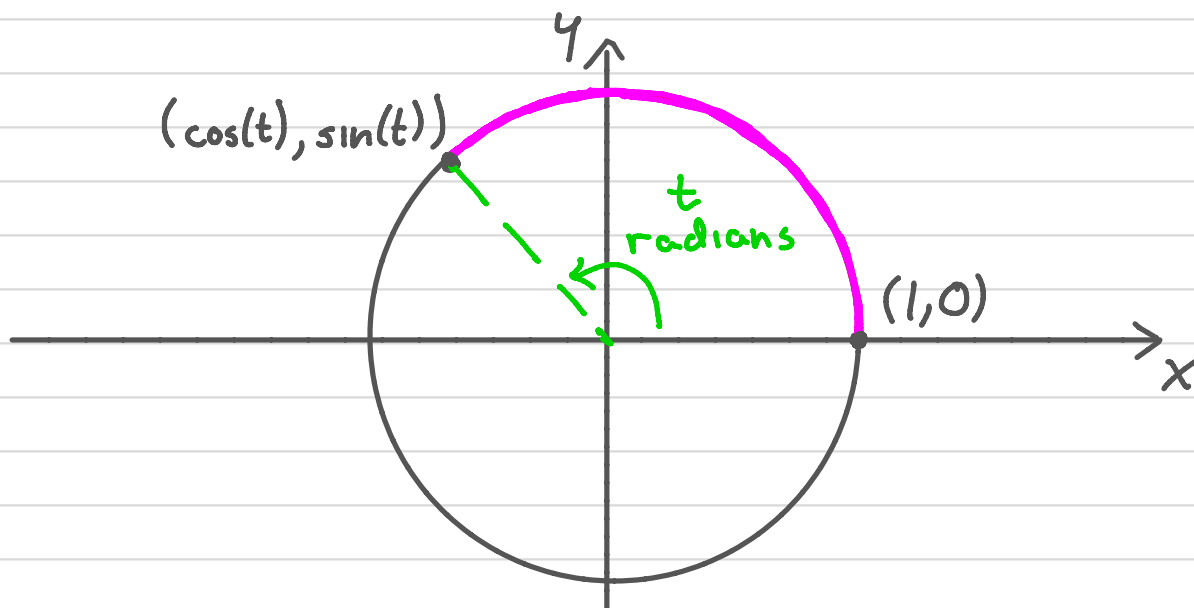$\left(-\frac{3\pi}{4}, \sin\left(-\frac{3\pi}{4}\right)\right)$

The units for t are <u>radians</u>, where 1 degree = $\frac{\pi}{180}$ radians.
E.g.

$$30° = 30 \cdot \frac{\pi}{180} = \frac{\pi}{6} \text{ radians}, \quad -\frac{5\pi}{4} \text{ radians} = -\frac{5\pi}{4} \cdot \frac{180}{\pi} = -225°,$$

etc.

The graph of $y = \cos(t)$ looks the same, but shifted <u>left</u> by $\pi/2$ units.

The functions $y = \cos(t)$ and $y = \sin(t)$ are called <u>circular</u> functions: they arise as coordinates of a point moving around the circle $x^2 + y^2 = 1$, like this:

$(\cos(t), \sin(t))$

t radians

$(1, 0)$

B) Remark. In general, if $y = f(x)$, we say "y is a function of x."
As noted above, this means x determines y __uniquely__ (that is, unambiguously).

For example, take $y = x^2$. For any x, only one y is possible: namely, $y = x^2$. E.g. $x = 3$ gives $y = 3^2 = 9$, unambiguously.

But note that __starting__ with $y = 9$ does __not__ give x unambiguously: $x^2 = 9$ could mean $x = 3$ __or__ $x = -3$.

Moral: "y is a function of x" __need__ __not__ imply "x is a function of y."

[Still, sometimes it __does__ imply this: e.g. $F = \frac{9}{5} C + 32$ can be solved to give
$$C = \frac{5}{9}(F - 32): \text{ a unique } C \text{ for each } F.]$$

C) Chaining (composing) functions.

The __chain__, or __composition__, of two functions f and g is when the __output__ $g(x)$ from g is __input__ into f.

$$x \longrightarrow \boxed{g} \longrightarrow g(x) \longrightarrow \boxed{f} \longrightarrow f(g(x))$$

The chain of f and g, denoted $f(g(x))$

__Example.__
Let $f(x) = 3x + 4$, $g(x) = x^2$, $h(x) = \frac{1}{x}$, $j(x) = \cos(x)$.
Then:
$$f(g(x)) = f(x^2) = 3x^2 + 4$$
$$g(f(x)) = g(3x + 4) = (3x + 4)^2 = 9x^2 + 24x + 16$$
(Note: here, and in general, $f(g(x)) \neq g(f(x))$.)
$$g(g(x)) = g(x^2) = (x^2)^2 = x^4$$

$$j(g(x)) = j(x^2) = \cos(x^2)$$
$$g(j(x)) = g(\cos(x)) = (\cos(x))^2,$$

often denoted $\cos^2(x)$; not the same as $\cos(x^2)$

$$h(f(2)) = h(3(2)+4) = h(10) = {}^1\!/_{10},$$

$$j(h(f(x))) = j(h(3x+4))$$
$$= j\left(\frac{1}{3x+4}\right) = \cos\left(\frac{1}{3x+4}\right),$$

etc.