Week 1 - Wednesday, 1/16

MORE ON SIR.

I) Recall the SIR equations:

$$\text{(SIR)} \quad \boxed{\begin{aligned} S' &= -aSI \\ I' &= aSI - bI \\ R' &= bI \end{aligned}}$$

$a\ (>0)$: transmission coefficient. Units: $(\text{person}\cdot\text{day})^{-1}$.
$b\ (>0)$: recovery coefficient. Units: day^{-1}.

Note: S is decreasing (since $S' < 0$).
I can increase and/or decrease.
R is increasing (since $R' > 0$).

Now, let's use (SIR) for:

II) Prediction. HUGE IDEA:

if Q is any quantity varying with time, then, between any two instants (a "new" one and an "old" one), we have

(a) new Q = old $Q + \Delta Q$,

where ΔQ is the change in Q. Moreover, we have

(b) $\Delta Q = Q' \cdot \Delta t$ *

where Δt is elapsed time, and Q' is the rate of change of Q.
[(b) says: net change equals rate of change times elapsed time.]

III) Let's now use (a), (b), and (SIR), as follows:

<u>Example.</u> Suppose we know that

$$\underbrace{S(0)}=500, \quad I(0)=10,\ R(0)=0,$$
$$a=0.001,\ b=0.2.$$

(this denotes the <u>value</u> of S at time $t=0$)

(i) Predict $S(2)$, $I(2)$, $R(2)$.

<u>Solution</u>. Let's start with S. We have

$$\begin{aligned}
S(2) &= S(0)+\Delta S && \text{(by (a) above)}\\
&= S(0)+S'(0)\cdot\Delta t && \text{(by (b) above)}\\
&= S(0)+(-a\cdot S(0)\cdot I(0))\cdot\Delta t && \text{(by (SIR))}\\
&= 500+(-0.001\cdot 500\cdot 10)\cdot 2 && \text{(plug in values)}\\
&= 500-10=490.
\end{aligned}$$

We do R next ('cause it's easier than I):

$$\begin{aligned}
R(2) &= R(0)+\Delta R\\
&= R(0)+R'(0)\cdot\Delta t\\
&= R(0)+b\cdot I(0)\cdot\Delta t\\
&= 0+0.2\cdot 10\cdot 2=4.
\end{aligned}$$

To find $I(2)$, we recall that $S+I+R$ is constant, and initially equal to $500+10+0=510$, so

$$I(2)=510-S(2)-R(2)=510-490-4=16.$$

Summary: according to the above model,

$$\boxed{S(2)=490,\ I(2)=16,\ R(2)=4}$$

(ii) Use part (i) above to predict $S(4)$, $I(4)$, $R(4)$.

Solution. $S(4) = S(2) + \Delta S$
$= S(2) + S'(2)\Delta t$
$= S(2) + (-a \cdot S(2) \cdot I(2)) \cdot \Delta t$
$= 490 + (-0.001 \cdot 490 \cdot 16) \cdot 2$ } plug in the values computed in Part (i) above
$= 474.32$

Similarly one finds $R(4)$ and $I(4)$; the net result is

$$\boxed{S(4) = 474.32, \quad I(4) = 25.28, \quad R(4) = 10.4.}$$

* NOTE: the "=" in the equation
$\Delta Q = Q' \Delta t$
(see II(b) above) should really be "≈" (approximately equals). Why? Because Q itself changes with time. More on this idea soon.

IV) Threshold value of S.

Typically, S will decrease until it can no longer sustain growth in I. At this point, I peaks.
The value of S at which this happens is called the threshold value S_T (see picture below).

Question: can we compute S_T?
Answer: YES! How? By noting that "I peaks" means "I changes from increasing to decreasing," which means "I' changes from positive to negative." At that instant, we must have $I' = 0$.

In sum,

$$\boxed{S_T \text{ is the value of } S \text{ where } I' = 0.} \qquad (A)$$

Now by (SIR),

$$I' = aSI - bI,$$

so by (A),

$$aS_T I - bI = 0.$$

Factor:

$$I(aS_T - b) = 0.$$

Divide by I:

$$aS_T - b = 0.$$

Solve for S_T:

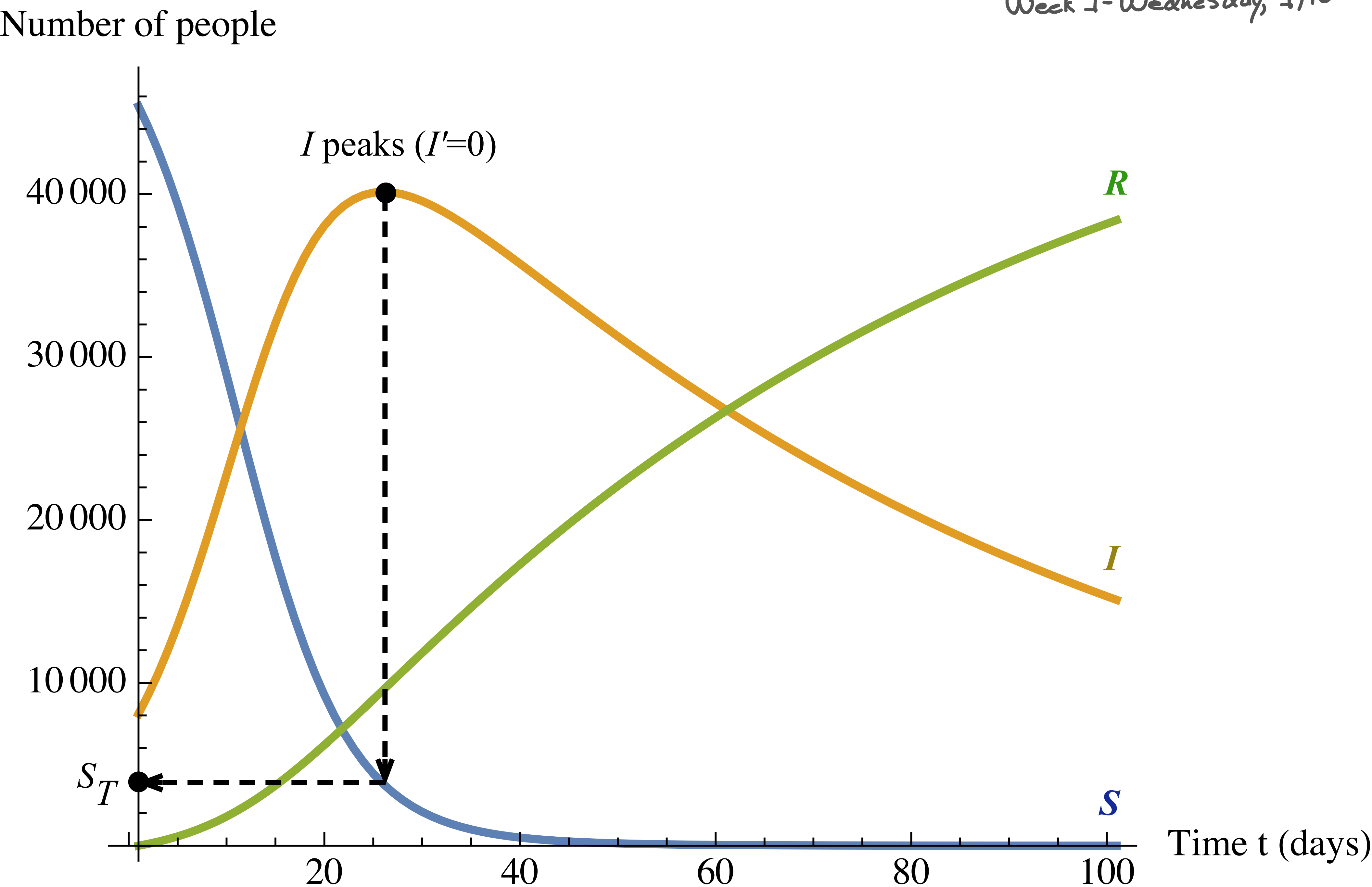
$$\boxed{S_T = b/a.}$$

formula for threshold value of S.

E.g. in our above example,

$$S_T = 0.2/0.001 = 200.$$

(Moral: often, interesting things happen when some rate of change equals zero!)


Number of people
I peaks (I′=0)
R
I
S
S_T
40 000
30 000
20 000
10 000
20
40
60
80
100
Time t (days)