Week 14 - Wednesday, 4/17

Separation of variables.

GOAL: to solve IVP's of the form

$$\frac{dy}{dx} = \text{something in } x \text{ times something in } y, \qquad y(x_0) = y_0.$$

Example 1. Solve the IVP

$$\frac{dy}{dx} = xy^2, \qquad y(1) = 3.$$

Solution. General strategy: We "separate, integrate, evaluate, solve:"

[Step 1: separate.] Get all y's (or whatever the dependent variable is) on the left; everything else on right:

$$\frac{dy}{y^2} = x\,dx.$$

[Step 2: integrate.] Take the indefinite integral on both sides: you only need one "+C."

$$\int \frac{dy}{y^2} = \int x\,dx$$

$$-y^{-1} = \frac{x^2}{2} + C.$$

[Step 3: evaluate.] Plug the IC into the step 2 result to solve for C; put this C back into step 2.

$$-3^{-1} = \frac{1^2}{2} + C$$

$$C = -3^{-1} - \frac{1}{2} = -\frac{1}{3} - \frac{1}{2} = \frac{-2-3}{6} = \frac{-5}{6}.$$

So by step 2,

$$-y^{-1} = \frac{x^2}{2} - \frac{5}{6}.$$

[Step 4: solve.] Solve for the dependent variable:

$$y^{-1} = -\left(\frac{x^2}{2} - \frac{5}{6}\right) = -\left(\frac{3x^2 - 5}{6}\right) = \frac{-3x^2+5}{6},$$

so

$$\boxed{y = \frac{6}{-3x^2+5}.}$$

Example 2. Solve the DE

$$\frac{dy}{dx} = \frac{\cos(x)}{e^y}.$$

Solution. Note: when there's no IC, we can skip step 3 ("evaluate").

[Step 1.] $$e^y\,dy = \cos(x)\,dx$$

[Step 2.] $$\int e^y\,dy = \int \cos(x)\,dx$$

$$e^y = \sin(x) + C.$$

[Step 4.] $$\ln(e^y) = \ln(\sin(x) + C).$$

$$\boxed{y = \ln(\sin(x) + C).}$$

Examples involving separation and substitution:

Example 3. Solve the IVP

$$\frac{dy}{dx} = e^{\sin(x)}\cos(x)\sqrt{y}, \quad y(0) = 4.$$

Solution.

$$\frac{dy}{\sqrt{y}} = e^{\sin(x)}\cos(x)\,dx$$

$$\int y^{-1/2}\,dy = \int e^{\sin(x)}\cos(x)\,dx$$

$u = \sin(x)$
$du = \cos(x)\,dx$

$$2y^{1/2} = \int e^{u}\,du = e^{u} + C$$

$$= e^{\sin(x)} + C.$$

Now $y(0) = 4$, so

$$2 \cdot 4^{1/2} = e^{\sin(0)} + C = 1 + C. \quad \text{So}$$

$$C = 2 \cdot 4^{1/2} - 1 = 2 \cdot 2 - 1 = 4 - 1 = 3.$$

So

$$2y^{1/2} = e^{\sin(x)} + 3$$

$$y^{1/2} = \frac{e^{\sin(x)} + 3}{2}$$

$$\boxed{y = \left(\frac{e^{\sin(x)} + 3}{2}\right)^2.}$$

Example 4.

$$\frac{dK}{dt} = \beta(\Gamma - K)$$

where β and Γ are constants, and $\Gamma > K$ always. Separate and integrate only.

Solution.

$$\frac{dK}{\Gamma - K} = \beta\,dt$$

$$\int \frac{dK}{\Gamma - K} = \int \beta\,dt = \beta t + C$$

$$\int -\frac{du}{u} = \beta t + C$$

integral in K:
$u = \Gamma - K$
$du = -dK$
$dK = -du$

$$-\int \frac{du}{u} = \beta t + C$$

$$-\ln(|u|) = \beta t + C$$

$$-\ln(|\Gamma - K|) = \beta t + C.$$

Now we've assumed $\Gamma > K$, so $\Gamma - K > 0$, so we don't need the absolute values, so

$$-\ln(\Gamma - K) = \beta t + C.$$