1. Evaluate the following *indefinite* integrals by substitution. The first one has been done for you, to remind you of the process.

(a)

$$\int \frac{x}{(x^2+1)^2}\,dx$$
$$= \int \frac{1}{u^2}\left(\frac{du}{2}\right) = \frac{1}{2}\int \frac{1}{u^2}\,du = -\frac{1}{2u} + C$$
$$= -\frac{1}{2(x^2+1)} + C.$$

$$u = x^2+1$$
$$du = 2x\,dx$$
$$\frac{du}{2} = x\,dx$$

(We've used the fact that $\int (1/u^2)\,du = \int u^{-2}\,du = -1 \cdot u^{-1} + C = -1/u + C$.)

(b) $\displaystyle\int \frac{12^{\ln(x)}}{x}\,dx$

$$\int \frac{12^{\ln(x)}}{x}\,dx = \int 12^u\,du$$
$$= \frac{12^u}{\ln(12)} + C = \frac{12^{\ln(x)}}{\ln(12)} + C.$$

$$u = \ln(x)$$
$$du = \frac{1}{x}\,dx$$

(c) $\displaystyle\int \sin\left(\frac{\pi x}{27}\right)dx$ (hint: try $u = \frac{\pi x}{27}$)

$$\int \sin\left(\frac{\pi x}{27}\right)dx = \int \sin(u)\left(\frac{27}{\pi}\,du\right)$$
$$= \frac{27}{\pi}\int \sin(u)\,du = \frac{27}{\pi}(-\cos(u)) + C$$
$$= -\frac{27}{\pi}\cos\left(\frac{\pi x}{27}\right) + C.$$

$$u = \frac{\pi x}{27}$$
$$du = \frac{\pi}{27}\,dx$$
$$dx = \frac{27}{\pi}\,du$$

(d) $\displaystyle\int \frac{\cos(x)}{1+\sin^2(x)}\,dx$ (hint: try $u = \sin(x)$)

$$\begin{aligned}&\int \frac{\cos(x)}{1+\sin^2(x)}\,dx\\ &= \int \frac{du}{1+u^2} = \arctan(u) + C\\ &= \arctan(\sin(x)) + C.\end{aligned}$$

$$\left| \begin{aligned} &u = \sin(x)\\ &du = \cos(x)\,dx \end{aligned}\right.$$

2. Evaluate the following *definite* integrals by substitution. Express each answer as a *rational number*, meaning either an integer, or an integer divided by another integer. Again, the first one has been done for you, to remind you of the process.

(a)

$$\begin{aligned}&\int_0^2 \frac{x^2}{(x^3+1)^2}\,dx\\ &= \int_1^9 \frac{1}{u^2}\left(\frac{du}{3}\right) = -\frac{1}{3u}\bigg|_1^9\\ &= -\frac{1}{27} + \frac{1}{3} = \frac{8}{27}.\end{aligned}$$

$$\left| \begin{aligned} &u = x^3+1\\ &du = 3x^2\,dx\\ &\frac{du}{3} = x^2\,dx\\ &\text{when } x = 0,\ u = 0^3+1 = 1\\ &\text{when } x = 2,\ u = 2^3+1 = 9 \end{aligned}\right.$$

(We've used the fact that $\int (1/u^2)\,du = \int u^{-2}\,du = -1 \cdot u^{-1} + C = -1/u + C$.)

(b) $\displaystyle\int_1^{e^2} \frac{\ln(x)}{x}\,dx$

$$\begin{aligned}&\int \frac{\ln(x)}{x}\,dx = \int_0^2 u\,du\\ &= \frac{u^2}{2}\bigg|_0^2 = 2.\end{aligned}$$

$$\left| \begin{aligned} &u = \ln(x)\\ &du = \frac{1}{x}\,dx\\ &\text{when } x = 1,\ u = \ln(1) = 0\\ &\text{when } x = e^2,\ u = \ln(e^2) = 2 \end{aligned}\right.$$

(c) $\displaystyle\int_0^{\sqrt{\pi/2}} z\cos(z^2)\,dz$

$$\int_0^{\sqrt{\pi/2}} z\cos(z^2)\,dz$$
$$= \int_0^{\pi/2} \cos(u)\left(\frac{du}{2}\right) = \frac{1}{2}\int_0^{\pi/2} \cos(u)\,du$$
$$= \left.\frac{\sin(u)}{2}\right|_0^{\pi/2} = \frac{1}{2} - \frac{0}{2} = \frac{1}{2}.$$

$u = z^2$

$du = 2z\,dz$

$\dfrac{du}{2} = z\,dz$

when $z = 0,\ u = 0^2 = 0$

when $z = \sqrt{\pi/2},\ u = (\sqrt{\pi/2})^2 = \pi/2$

(d) $\displaystyle\int_{\ln(\ln(2))}^{\ln(\ln(6))} e^y \cdot e^{e^y}\,dy$

$$\int_{\ln(\ln(2))}^{\ln(\ln(6))} e^y \cdot e^{e^y}\,dy$$
$$= \int_{\ln(2)}^{\ln(6)} e^u\,du = e^u\Big|_{\ln(2)}^{\ln(6)}$$
$$= e^{\ln(6)} - e^{\ln(2)} = 6 - 2 = 4.$$

$u = e^y$

$du = e^y\,dy$

when $y = \ln(\ln(2)),\ u = e^{\ln(\ln(2))} = \ln(2)$

when $y = \ln(\ln(6)),\ u = e^{\ln(\ln(6))} = \ln(6)$

3. Solve the initial value problem

$$\frac{dy}{dx} = xe^{x^2}, \qquad y(0) = 5.$$

$$y = \int xe^{x^2}\,dx$$
$$= \int e^u \left(\frac{du}{2}\right) = \frac{1}{2}\int e^u\,du = \frac{1}{2}e^u + C$$
$$= \frac{1}{2}e^{x^2} + C.$$

$$\left| \begin{array}{l} u = x^2 \\ du = 2x\,dx \\ \dfrac{du}{2} = x\,dx \end{array} \right.$$

Then plugging in $y(0) = 5$ gives

$$5 = \frac{1}{2}e^{0^2} + C = \frac{1}{2} + C,$$

so

$$C = 5 - \frac{1}{2} = \frac{9}{2},$$

so

$$y = \frac{1}{2}e^{x^2} + \frac{9}{2}.$$