Week 14 - Monday, 4/15

Substitution, continued.

Recall: if an integral contains a quantity $g(x)$ whose derivative $g'(x)$ is also present, then putting $u = g(x)$ can simplify things.

(A) Indefinite integrals (review).

Example 1.

$$\int e^x \cos(e^x)\,dx = \int \cos(u)\,du = \sin(u) + C = \sin(e^x) + C.$$

$u = e^x$
$du = e^x\,dx$

Check: $\frac{d}{dx}[\sin(e^x) + C] = \cos(e^x) \cdot \frac{d}{dx}[e^x] + 0$
$= e^x \cos(e^x).$ ✓

Example 2.

$$\int t^3 (t^4+5)^7\,dt = \int u^7 \left(\frac{du}{4}\right) = \frac{1}{4}\int u^7\,du = \frac{1}{4} \cdot \frac{u^8}{8} + C = \frac{u^8}{32} + C = \frac{(t^4+5)^8}{32} + C.$$

$u = t^4 + 5$
$du = 4t^3\,dt$
$t^3\,dt = \frac{du}{4}$

Example 3: the general idea.

$$\int f(g(x))\,g'(x)\,dx = \int f(u)\,du,$$

$u = g(x)$
$du = g'(x)\,dx$

which is simpler than the integral we started with.

(B). Definite integrals.

Proceed as above, BUT: also change your limits of integration!!

Example 4.

$$\int_3^5 \frac{1}{x(\ln(x))^2}\,dx$$

$$= \int_{\ln(3)}^{\ln(5)} \frac{1}{u^2}\,du = \int_{\ln(3)}^{\ln(5)} u^{-2}\,du$$

$$= -u^{-1}\Big|_{\ln(3)}^{\ln(5)}$$

$$= \frac{-1}{u}\Big|_{\ln(3)}^{\ln(5)} = \frac{-1}{\ln(5)} + \frac{1}{\ln(3)}.$$

$u = \ln(x)$
$du = \frac{1}{x}\,dx$
when $x = 3$, $u = \ln(3)$
when $x = 5$, $u = \ln(5)$

Example 5.

$$\int_0^2 x e^{-x^2}\,dx$$

$$= \int_0^{-4} e^u\left(-\frac{du}{2}\right) = \frac{-1}{2}\int_0^{-4} e^u\,du$$

$$= \frac{-1}{2}e^u\Big|_0^{-4} = \frac{-1}{2}e^{-4} + \frac{1}{2}e^0 = \frac{-e^{-4}+1}{2}.$$

$u = -x^2$
$du = -2x\,dx$
$x\,dx = -\frac{du}{2}$
when $x = 0$, $u = -0^2 = 0$
when $x = 2$, $u = -2^2 = -4$

Example 6.

$$\int_{\pi/2}^{\pi} \cos(\pi\sin(x))\cos(x)\,dx$$

$$= \int_1^0 \cos(\pi u)\,du = \frac{\sin(\pi u)}{\pi}\Big|_1^0$$

$$= \frac{1}{\pi}(\sin(\pi\cdot 0) - \sin(\pi\cdot 1))$$

$$= \frac{1}{\pi}(0-0) = 0.$$

$u = \sin(x)$
$du = \cos(x)\,dx$

when $x = \pi/2$, $u = \sin(\pi/2) = 1$
when $x = \pi$, $u = \sin(\pi) = 0$

(C). IVP's.

We've seen: to solve the IVP

$$\frac{dy}{dx} = f(x), \qquad y(0) = y_0,$$

we:

(1) Integrate:

$$y = \int f(x)\,dx = F(x) + C \qquad (*)$$

for some function $F(x)$ and constant C;

(2) Plug the IC into $(*)$ to get

$$y_0 = y(0) = F(0) + C,$$

then solve this for C;

(3) Plug the C you got in (2) back into $(*)$, to get your complete answer.

SOMETIMES, step (1) may require a substitution.

Example 7. Solve the IVP

$$\frac{dy}{dx} = \frac{(\ln(x)+2)^3}{x}, \qquad y(1) = 7.$$

Solution.

(1) $$y = \int \frac{(\ln(x)+2)^3}{x}\,dx \qquad \left|\; \begin{aligned} u &= \ln(x)+2 \\ du &= \frac{1}{x}\,dx \end{aligned}\right.$$

$$= \int u^3\,du$$

$$= \frac{u^4}{4} + C = \frac{(\ln(x)+2)^4}{4} + C.$$

(2) By (1) and our initial condition,

$$7 = y(1) = \frac{(\ln(1)+2)^4}{4} + C = \frac{(0+2)^4}{4} + C = \frac{16}{4} + C = 4 + C.$$

So $C = 7 - 4 = 3$.

(3) By (1) and (2),

$$y = \frac{(\ln(x)+2)^4}{4} + 3.$$