Week 14 - Friday, 4/19

Separation of variables, continued.

Recall that, to solve IVP's like

$$\frac{dy}{dx} = f(x)g(y), \qquad y(x_0) = y_0, \qquad (*)$$

we use a "separate, integrate, evaluate, solve" strategy.
(If there's no IC, we skip the "evaluate" step.)

I) Application to "old school" DE's, of the form
$\frac{dy}{dx} = f(x)$ (where the $g(y)$ in $(*)$ just equals 1).
We've done these before by just writing $y = \int f(x)dx$, but our separation strategy works here too!

Example 1 Solve
$$\frac{dy}{dx} = 6x^2, \qquad y(2) = 4.$$

Solution.

Separate: $dy = 6x^2 dx$.

Integrate: $\int dy = \int 6x^2 dx$

$y = 2x^3 + C$

Evaluate: $y(2) = 4$ so

$4 = 2 \cdot 2^3 + C = 16 + C$

$C = 4 - 16 = -12$

So $\boxed{y = 2x^3 - 12.}$

Example 2: Newton's Law of Cooling.
(Note: this is very similar to Example 4 of Wednesday 4/17, and to "Diffusion across a membrane" in HW 11.)

An object of temperature Q, in a room of constant temperature $A<Q$, cools at a rate proportional to the temperature difference $Q-A$. That is,

$$\frac{dQ}{dt} = -k(Q-A) \qquad (k>0).$$

(a) Find a formula for Q in terms of k and A.

(b) Find an exact formula for Q if
(i) $k = 0.1\ ^\circ C/(\text{min} \cdot {}^\circ C)$ and $A = 20\ ^\circ C$
(ii) $Q(0) = 90^\circ C$.

Solution.
Separate:

$$\frac{dQ}{Q-A} = -k\,dt$$

Integrate:

$$\int \frac{dQ}{Q-A} = \int -k\,dt$$

$$\int \frac{du}{u} = -kt+C$$

$$\ln(|u|) = -kt+C$$

$$\ln(|Q-A|) = -kt+C$$

integral in Q:
$u = Q-A$
$du = dQ$

Solve: Since a cooling object will never cool to below room temperature, we have $Q>A$ always, so $Q-A>0$ always, so $|Q-A| = Q-A$ always, so the above result gives

$\ln(Q-A) = -kt+C.$

Then

$$e^{\ln(Q-A)} = e^{-kt+C}$$
$$Q-A = e^{-kt+C}$$
$$Q = Ae^{-kt+c}.$$

Clean up: note that $e^{-kt+c} = e^{-kt}e^{c}$. Now e^{c} is just a constant, call it M, so we get

$$\boxed{Q = A + Me^{-kt}}$$

(A, M, k are positive constants).

(b) If $k=0.1$ and $A=20$, we get

$$Q = 20 + Me^{-0.1t}.$$

Plugging in $Q(0)=90$ gives

$$90 = 20 + Me^{-0.1\cdot 0} = M\cdot 1 = M,$$

so $M = 90-20 = 70$, so

$$\boxed{Q = 20 + 70e^{-0.1t}.}$$