A useful integration technique:

Integration by substitution

(= the chain rule in reverse, sort of).

Example 1. Find $\int 2x\cos(x^2)\,dx$.

Solution.
The key is that the integral contains something — namely, x^2 — whose derivative — namely, $2x$ — is also present.

The trick is to call that "something" u:
$$u = x^2.$$
Then
$$\frac{du}{dx} = 2x,$$
so
$$du = 2x\,dx.^{*}$$
So
$$\int 2x\cos(x^2)\,dx = \int \cos(u)\,du = \sin(u) + C = \sin(x^2) + C.$$
↑ plug back in for u

Check:
$$\frac{d}{dx}\left[\sin(x^2)+C\right] = \cos(x^2)\cdot\frac{d}{dx}\left[x^2\right] + 0$$
$$= \cos(x^2)\cdot 2x = 2x\cos(x^2). \checkmark$$

* Technically, $\frac{du}{dx}$ isn't a fraction, so why can we "multiply $\frac{du}{dx} = 2x$ by dx"?
Why? Because it works! The "check" illustrates this (and shows how the chain rule figures in).

Example 2. (Note the work "in the margin.")

$$\int 5x^4 (2+x^5)^{26}\,dx$$

$$= \int u^{26}\,du = \frac{u^{27}}{27} + C$$

$$= \frac{(2+x^5)^{27}}{27} + C.$$

[DIY: check.]

$u = 2 + x^5$ (choose a "u" whose derivative also appears)

$\frac{du}{dx} = 5x^4$ (differentiate)

$du = 5x^4\,dx$ (multiply by dx)

Example 3.

$$\int e^{\sin(x)} \cos(x)\,dx$$

$$= \int e^u\,du$$

$$= e^u + C = e^{\sin(x)} + C.$$

Check:

$$\frac{d}{dx}\left[e^{\sin(x)} + C\right] = e^{\sin(x)} \cdot \cos(x) + 0$$

$$= e^{\sin(x)} \cos(x) \checkmark$$

$u = \sin(x)$

$\frac{du}{dx} = \cos(x)$

$du = \cos(x)\,dx$

Example 4.

$$\int \frac{e^x}{1+(e^x+4)^2}\,dx$$

$$= \int \frac{1}{1+u^2}\,du = \arctan(u) + C$$

$$= \arctan(e^x + 4) + C.$$

$u = e^x + 4$

$du = e^x\,dx$ *

* We went right from $u = \ldots$ to $du = \ldots$, skipping the step $\frac{du}{dx} = \ldots$ in between.

Example 5.

$$\int \frac{\sin(\ln(z))}{z}\,dz$$

$$= \int \sin(u)\,du = -\cos(u) + C$$
$$= -\cos(\ln(z)) + C.$$

$u = \ln(z)$
$du = \frac{1}{z}\,dz$

Example 6.

$$\int x\sin(x^2+1)\,dx$$
$$= \int \sin(u)\cdot\left(\frac{du}{2}\right)$$
$$= \frac{1}{2}\int \sin(u)\,du = -\frac{1}{2}\cos(u) + C$$
$$= -\frac{1}{2}\cos(x^2) + C.$$

$u = x^2 + 1$
$du = 2x\,dx$ *
$\frac{du}{2} = x\,dx$

our substitution gives a $2x\,dx$, but our integral has only $x\,dx$, so divide by 2

* If your u-substitution gives your du an unwanted constant factor, like 2, just divide by this factor.

Example 7.

$$\int x^2 e^{x^3}\,dx$$
$$= \int e^u\left(\frac{du}{3}\right) = \frac{1}{3}\int e^u\,du$$
$$= \frac{1}{3}e^u + C$$
$$= \frac{1}{3}e^{x^3} + C.$$

$u = x^3$
$du = 3x^2\,dx$
$\frac{du}{3} = x^2\,dx$

Example 8.

$$\int e^{7y}\,dy = \int e^u\left(\frac{du}{7}\right)$$
$$= \frac{1}{7}\int e^u\,du = \frac{1}{7}e^u + C = \frac{1}{7}e^{7y} + C.$$

$u = 7y$
$du = 7\,dy$
$\frac{du}{7} = dy$

Example 9. $\int \cos(x^4)\,dx = ?$

We could try $u = x^4$; but then $du = 4x^3 dx$, and what do we do with the x^3?

Fact: sometimes substitution fails. (Also: $\cos(x^4)$ has no nice antiderivative.)

Next time: substitution in definite integrals.