

A useful integration technique:
Integration by substitution.

Example 1. Find $\int 2x \cos(x^2) dx$.

Solution.

The key is that the integral contains something — namely, x^2 — whose derivative — namely, $2x$ — is also present.

The trick is to call that "something" u :
 $u = x^2$.

Then $\frac{du}{dx} = 2x,$

so $du = 2x dx.$ *

So $\int \underbrace{2x}_{\text{green}} \underbrace{\cos(x^2)}_{\text{pink}} \underbrace{dx}_{\text{green}} = \int \underbrace{\cos(u)}_{\text{pink}} \underbrace{du}_{\text{green}} = \sin(u) + C = \sin(x^2) + C.$
↑ plug back in for u

Check:

$$\begin{aligned} \frac{d}{dx} [\sin(x^2) + C] &= \cos(x^2) \cdot \frac{d}{dx} [x^2] + 0 \\ &= \cos(x^2) \cdot 2x = 2x \cos(x^2). \quad \checkmark \end{aligned}$$

* This works if we imagine that du/dx really is the quotient of infinitesimals du and dx .

Example 2. (Note the work "in the margin.")

$$\begin{aligned} \int 5x^4(2+x^5)^{26} dx \\ = \int u^{26} du = \frac{u^{27}}{27} + C \\ = \frac{(2+x^5)^{27}}{27} + C. \end{aligned}$$

[DIY: check.]

(choose a "u" whose derivative also appears)

$$\begin{aligned} u &= 2+x^5 \\ \frac{du}{dx} &= 5x^4 \quad (\text{differentiate}) \\ du &= 5x^4 dx \quad (\text{multiply by } dx) \end{aligned}$$

Example 3.

$$\begin{aligned} \int e^{\sin(x)} \cos(x) dx \\ = \int e^u du \\ = e^u + C = e^{\sin(x)} + C. \end{aligned}$$

$$\begin{aligned} u &= \sin(x) \\ \frac{du}{dx} &= \cos(x) \\ du &= \cos(x) dx \end{aligned}$$

Check:

$$\frac{d}{dx} [e^{\sin(x)} + C] = e^{\sin(x)} \cdot \cos(x) + 0 = e^{\sin(x)} \cos(x) \checkmark$$

Example 4.

$$\begin{aligned} \int \frac{e^x}{1+(e^x+4)^2} dx \\ = \int \frac{1}{1+u^2} du = \arctan(u) + C \\ = \arctan(e^x+4) + C. \end{aligned}$$

$$\begin{aligned} u &= e^x+4 \\ du &= e^x dx \end{aligned} *$$

* We went right from $u = \dots$ to $du = \dots$, skipping the step $\frac{du}{dx} = \dots$ in between.

Example 5.

$$\int \frac{\sin(\ln(z))}{z} dz$$

$$= \int \sin(u) du = -\cos(u) + C$$

$$= -\cos(\ln(z)) + C.$$

$$\left| \begin{array}{l} u = \ln(z) \\ du = \frac{1}{z} dz \end{array} \right.$$

Example 6.

$$\int x \sin(x^2 + 1) dx$$

$$= \int \sin(u) \cdot \left(\frac{du}{2} \right)$$

$$= \frac{1}{2} \int \sin(u) du = -\frac{1}{2} \cos(u) + C$$

$$= -\frac{1}{2} \cos(x^2 + 1) + C.$$

$$u = x^2 + 1$$

$$du = 2x dx$$

$$\frac{du}{2} = x dx$$

our substitution gives a $2x dx$, but our integral has only $x dx$, so divide by 2

* If your u -substitution gives your du an unwanted constant factor, like 2, just divide by this factor.

Example 7.

$$\int x^2 e^{x^3} dx$$

$$= \int e^u \left(\frac{du}{3} \right) = \frac{1}{3} \int e^u du$$

$$= \frac{1}{3} e^u + C = \frac{1}{3} e^{x^3} + C.$$

$$\left| \begin{array}{l} u = x^3 \\ du = 3x^2 dx \\ \frac{du}{3} = x^2 dx \end{array} \right.$$

Example 8.

$$\int e^{7y} dy = \int e^u \left(\frac{du}{7} \right)$$

$$= \frac{1}{7} \int e^u du = \frac{1}{7} e^u + C = \frac{1}{7} e^{7y} + C.$$

$$\left| \begin{array}{l} u = 7y \\ du = 7 dy \\ \frac{du}{7} = dy \end{array} \right.$$

Example 9.

$$\int \cos(x^4) dx = ?$$

We could try $u = x^4$, but then $du = 4x^3 dx$, and what do we do with the x^3 ?

Fact: sometimes substitution fails. (Also: $\cos(x^4)$ has no nice antiderivative.)

Next time: substitution in definite integrals.