Week 13 - Monday, 11/16

Antiderivatives and initial value problems (IVP's).

Idea: to solve an IVP of the form

$$\underbrace{\frac{dy}{dx} = f(x),}_{\text{differential equation (DE)}} \qquad \underbrace{y(0) = y_0,}_{\text{initial condition (IC)}}$$

do this:

Step 1. Antidifferentiate (using an indefinite integral) to find the most general solution, with a $+C$, to the DE;

Step 2. Use the IC to solve for C. Then put this C back into the answer from Step 1 to finish.

Examples.

1) Solve the IVP

$$\frac{dy}{dx} = x - \sin(2x), \qquad y(0) = 3.$$

Solution.

First we solve the DE:

$$y = \int (x - \sin(2x))\,dx = \frac{x^2}{2} + \frac{\cos(2x)}{2} + C.$$

Next, plug the IC $y(0) = 3$ into the above y:

$$3 = \frac{0^2}{2} + \frac{\cos(2 \cdot 0)}{2} + C = \frac{1}{2} + C.$$

Solve:

$$C = 3 - \tfrac{1}{2} = \tfrac{5}{2}.$$

Finally, put this C back into the above y:

$$\boxed{y = \frac{x^2 + \cos(2x) + 5}{2}.}$$

Example 2.

Find a function $F(q)$ satisfying

$$F'(q) = 5q^4 - \frac{2}{q}, \qquad F(-1) = -2.$$

Solution.

First, we solve the DE:

$$F(q) = \int F'(q)\,dq = \int \left(5q^4 - \frac{2}{q}\right) dq = q^5 - 2\ln(|q|) + C.$$

Now plug in the IC:

$$-2 = F(-1) = (-1)^5 - 2\ln(|-1|) + C$$
$$= -1 - 2\ln(1) + C$$
$$= -1 - 2\cdot 0 + C = -1 + C.$$

So

$$C = -2 + 1 = -1.$$

Finally, put this back into $F(q)$ from above:

$$\boxed{F(q) = q^5 - 2\ln(|q|) - 1.}$$