Logistic growth.

Example: solve the logistic growth IVP

$$\frac{dP}{dt} = kP\left(1-\frac{P}{b}\right), \qquad P(0)=P_0.$$

Assume $P_0 < b$.

Solution.

[Step 0: some algebra.]

Note that

$$kP(1-P/b) = kP(P\cdot(P^{-1}-b^{-1})),$$
$$= kP^2(P^{-1}-b^{-1}),$$

so the logistic growth DE reads

$$\frac{dP}{dt} = kP^2(P^{-1}-b^{-1}).$$

[STEP 1: separate.]

So we get

$$\frac{dP}{P^2(P^{-1}-b^{-1})} = k\,dt.$$

[STEP 2: integrate.]

$$\int \frac{dP}{P^2(P^{-1}-b^{-1})} = \int k\,dt$$

$$\int \frac{-du}{u} = \int k\,dt$$

$$-\ln(|u|) = kt + C$$

integral in P:

$u = P^{-1}-b^{-1}$

$du = -P^{-2}\,dP$

$-du = \dfrac{dP}{P^2}$

$$-\ln(|P^{-1}-b^{-1}|) = kt + C. \qquad (*)$$

[Step 3: evaluate.]

$P(0) = P_0$, so $(*)$ gives

$$-\ln(|P_0^{-1}-b^{-1}|) = 0\cdot t + C = C.$$

Putting this value of C back into $(*)$ gives

$$-\ln(|P^{-1}-b^{-1}|) = kt - \ln(|P_0^{-1}-b^{-1}|). \qquad (*')$$

But now note: Since $P_0 < b$ by assumption, we have $P_0^{-1} > b^{-1}$, so $P_0^{-1}-b^{-1} > 0$. Also, since the initial population $P_0 = P(0)$ is $< b$, one can show that P is <u>always</u> $< b$. So $P^{-1}-b^{-1} > 0$ always.

Since both quantities inside the absolute values in $(*')$ are >0, we may drop the absolute values, to get

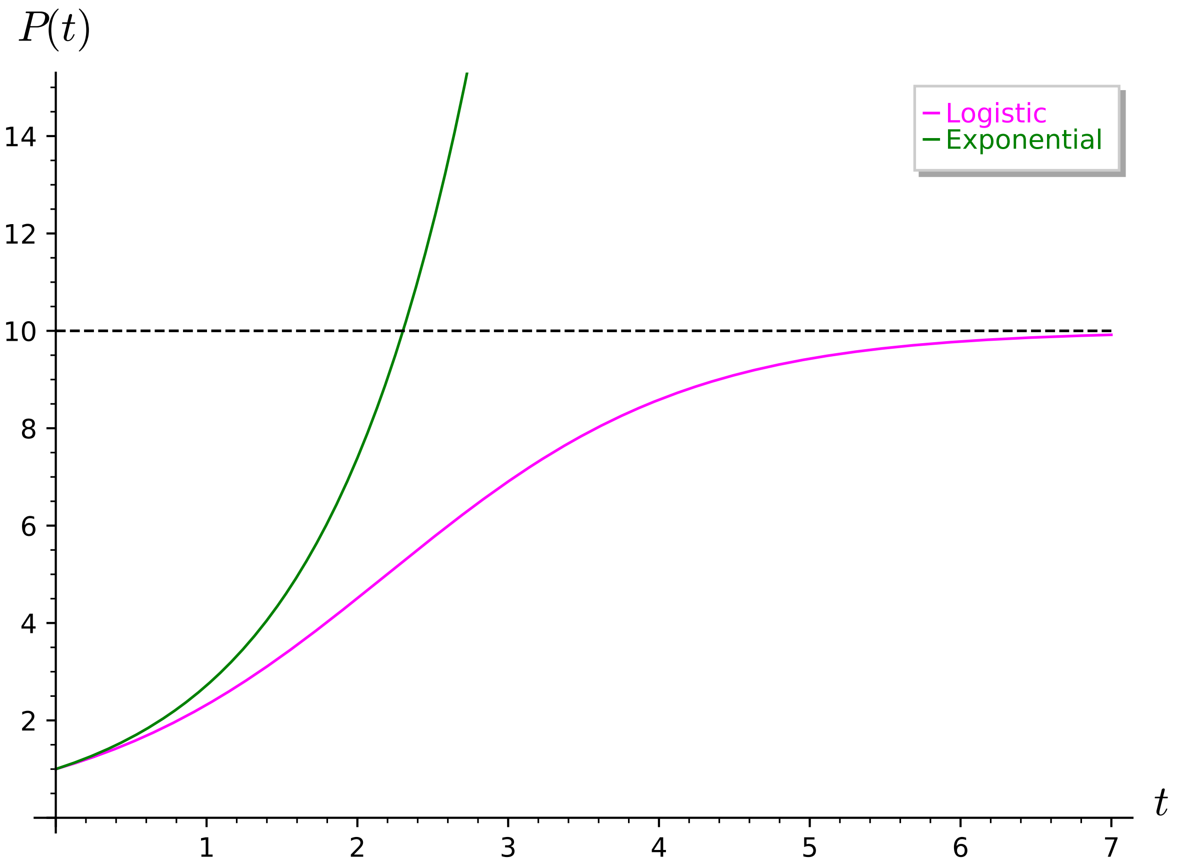
$$-\ln(P^{-1}-b^{-1}) = kt - \ln(P_0^{-1}-b^{-1}).$$

[Step 4: solve.]

Sage tells us that the above equation gives

$$P = \frac{P_0 b e^{kt}}{b + P_0(e^{kt}-1)}.$$

The graph of P (compared to the graph of the corresponding exponential function $P = P_0 e^{kt}$) looks like this:



$P(t)$
14
12
10
8
6
4
2
1
2
3
4
5
6
7
t
Logistic
Exponential