Integrals: the story so far.

I. Summary.

Given a function $f(x)$, let $F(x)$ be an antiderivative for $f(x)$, meaning

$$F'(x) = f(x).$$

Then:

(A) The definite integral $\int_a^b f(x)\,dx$, which is the signed area between $f(x)$ and the interval $[a,b]$ on the x-axis, is given by FTC:

$$\int_a^b f(x)\,dx = F(x)\Big|_a^b = F(b) - F(a).$$

(B) The indefinite integral $\int f(x)\,dx$, which is the set of all antiderivatives of $f(x)$, is given by the COOL FACT of last time:

$$\int f(x)\,dx = F(x) + C$$

where C denotes an arbitrary constant.

II. Short table of indefinite integrals. (Here, a, b, and p are constants.)

$f(x)$	$\int f(x)\,dx$
x^p	$\dfrac{x^{p+1}}{p+1} + C \quad (p \neq -1)$
$1/x$	$\ln(\lvert x\rvert) + C$
$\sin(ax)$	$\dfrac{-\cos(ax)}{a} + C \quad (a \neq 0)$
$\cos(ax)$	$\dfrac{\sin(ax)}{a} + C \quad (a \neq 0)$

(continued)

$f(x)$	$\int f(x)\,dx$
e^{ax}	$\dfrac{e^{ax}}{a} + C \quad (a \neq 0)$
b^x	$\dfrac{b^x}{\ln(b)} + C \quad (b > 0)$
$\dfrac{1}{1+x^2}$	$\arctan(x) + C$

Note that we can <u>check</u> the above table by differentiation:
e.g.

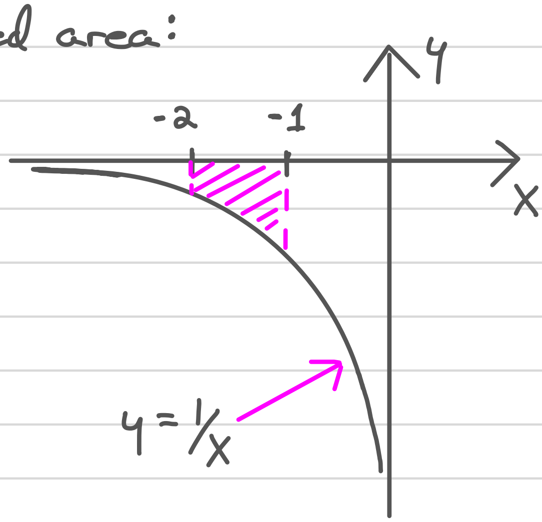
$$\frac{d}{dx}\left[\frac{\sin(ax)}{a}+C\right] = \frac{1}{a}\frac{d}{dx}[\sin(ax)] + \frac{d}{dx}[C]$$
$$= \frac{1}{a}\cdot\cos(ax)\cdot a + 0 = \cos(ax). \checkmark$$

III. <u>Examples</u>: use the above table to find the indicated quantities.

(1) $\int \left(\frac{3}{x^2} + 3x^2 - 3\sqrt{x} + \frac{3}{\sqrt{x}}\right) dx$

(2) $\int_0^{\pi/2} (\cos(2x) - 2\sin(x))\, dx$

(3) The signed area:



(4) The solution to the IVP

$$\frac{dy}{dx} = \frac{1}{1+x^2} + e^{3x}, \qquad y(0) = 2.$$

Solutions

(1) $\int \left(\frac{3}{x^2} + 3x^2 - 3\sqrt{x} + \frac{3}{\sqrt{x}} \right) dx = \int (3x^{-2} + 3x^2 - 3x^{1/2} + 3x^{-1/2}) dx$

$= \frac{3x^{-1}}{-1} + \frac{3x^3}{3} - \frac{3x^{3/2}}{3/2} + \frac{3x^{1/2}}{1/2} + C = -3x^{-1} + x^3 - 2x^{3/2} + 6x^{1/2} + C.$

(2) $\int_0^{\pi/2} (\cos(2x) - 2\sin(x)) dx = \left(\frac{\sin(2x)}{2} + 2\cos(x) \right) \Big|_0^{\pi/2}$

$= \left(\frac{\sin(2 \cdot \pi/2)}{2} + 2\cos(\pi/2) \right) - \left(\frac{\sin(2 \cdot 0)}{2} + 2\cos(0) \right)$

$= (0/2 + 2 \cdot 0) - (0/2 + 2) = -2.$

(3) The signed area is

$\int_{-2}^{-1} \frac{1}{x} dx = \ln(|x|) \Big|_{-2}^{-1} = \ln(|-1|) - \ln(|-2|)$

$= \ln(1) - \ln(2) = 0 - \ln(2) = -\ln(2) \approx -0.6931.$

Note: without the absolute values, we'd be trying to evaluate things like $\ln(-2)$, which doesn't make sense.

(4) First, to get y from dy/dx, we antidifferentiate:

$y = \int \left(\frac{1}{1+x^2} + e^{3x} \right) dx = \arctan(x) + \frac{e^{3x}}{3} + C. \qquad (*)$

Next, plug the initial condition into $(*)$, to find C:

$2 = y(0) = \arctan(0) + \frac{e^{3 \cdot 0}}{3} + C = 0 + \frac{1}{3} + C = \frac{1}{3} + C.$

So

$$C = 2 - \tfrac{1}{3} = \tfrac{5}{3}.$$

So by $(*)$,

$$y = \arctan(x) + \frac{e^{3x}}{3} + \frac{5}{3}.$$