Week 12 - Tuesday, 4/2

Antiderivatives

Recall the Fundamental Theorem of Calculus (FTC):

$$\int_a^b f(x)\,dx = F(x)\Big|_a^b = F(b) - F(a),$$

where $F(x)$ is any antiderivative of $f(x)$, meaning $F'(x) = f(x)$.

Example 1.

(a) Find two antiderivatives $F(x)$ and $G(x)$ of $f(x) = x^2$.

(b) Evaluate $\int_1^4 x^2\,dx$ in two ways.

Solution.

(a) $F(x) = \frac{x^3}{3}$ works, since $F'(x) = \frac{1}{3} \cdot 3x^2 = x^2 = f(x)$. But so does $G(x) = \frac{x^3}{3} + 5$, since $G'(x) = \frac{1}{3} \cdot 3x^2 + 0 = x^2 = f(x)$.

(b) By FTC,

$$\int_1^4 x^2\,dx = \frac{x^3}{3}\Big|_1^4 = \frac{4^3}{3} - \frac{1^3}{3} = \frac{64-1}{3} = \frac{63}{3} = 21.$$

OR

$$\int_1^4 x^2\,dx = \left(\frac{x^3}{3} + 5\right)\Big|_1^4 = \left(\frac{4^3}{3} + \cancel{5}\right) - \left(\frac{1^3}{3} + \cancel{5}\right)$$

$$= \frac{4^3}{3} - \frac{1^3}{3} = \frac{64-1}{3} = \frac{63}{3} = 21.$$

COOL FACT:

in general, if $F(x)$ is one antiderivative of $f(x)$, then any antiderivative of $f(x)$ is of the form

$$F(x)+C$$

for some constant C.

Let's write

$$\int f(x)\,dx$$

"the indefinite integral of $f(x)\,dx$"

for the set of all antiderivatives of $f(x)$.

Examples: find these indefinite integrals and check your work.

2) $\int \cos(x)\,dx = \sin(x)+C.$

The COOL FACT says: to find $\int f(x)\,dx$,

a) first find some (any) antiderivative of $f(x)$;

b) then tack on the arbitrary constant "$+C$."

DON'T FORGET THIS STEP!!

Check:

$$\frac{d}{dx}[\sin(x)+C] = \cos(x)+0 = \cos(x) \checkmark$$

3) $\int \sin(x)\,dx = -\cos(x)+C.$

Check:

$$\frac{d}{dx}[-\cos(x)+C] = -(-\sin(x))+0 = \sin(x).$$

4) $\int \sin(2x)\,dx = \dfrac{-\cos(2x)}{2} + C.$

Check:
$$\frac{d}{dx}\left[\frac{-\cos(2x)}{2}+C\right] = \frac{-1}{2}(-\sin(2x)\cdot 2)+0 = \sin(2x). \checkmark$$

5) $\int e^{7y}\,dy = \dfrac{e^{7y}}{7} + C.$

Check: $\dfrac{d}{dy}\left[\dfrac{e^{7y}}{7}+C\right] = \dfrac{1}{7}\cdot e^{7y}\cdot 7+0 = e^{7y}. \checkmark$

6) For any $b>0$,
$$\int b^x\,dx = \frac{b^x}{\ln(b)} + C.$$
Check: $\dfrac{d}{dx}\left[\dfrac{b^x}{\ln(b)}+C\right] = \dfrac{1}{\ln(b)}\cdot \ln(b)\, b^x + 0 = b^x. \checkmark$

7) $\int (3t^2 - 7t + 2)\,dt = t^3 - \frac{7}{2}t^2 + 2t + C.$

Check: DIY.

Remark. Recall that the FTC says: for definite integrals, proceed essentially as above, but replace the "+C" with "$\big|_a^b$".
For example:

8) $\displaystyle\int_0^{\pi/4} \sin(2x)\,dx = \frac{-\cos(2x)}{2}\Big|_0^{\pi/4} = \frac{-\cos(2\cdot\pi/4)}{2} - \left(\frac{-\cos(2\cdot 0)}{2}\right) = \frac{-0}{2} + \frac{1}{2} = \frac{1}{2}.$

9) $\displaystyle\int_0^1 e^{7y}\,dy = \frac{e^{7y}}{7}\Big|_0^1 = \frac{e^{7\cdot 1}}{7} - \frac{e^{7\cdot 0}}{7} = \frac{e^7 - e^0}{7} = \frac{e^7 - 1}{7}.$

Finally, an IVP:

10) Find a function $F(z)$ with $F'(z) = 4z+1$ and $F(1) = -4$.

Solution.

First, to find $F(z)$ given $F'(z)$, we antidifferentiate (do an indefinite integral):

$$F(z) = \int F'(z)\,dz = \int (4z+1)\,dz = 2z^2 + z + C. \qquad (*)$$

Next, to find C, we plug the IC $F(1) = -4$ into $(*)$:

$$-4 = F(1) = 2(1)^2 + 1 + C = 3 + C$$

Solve for C:

$$C = -4 - 3 = -7.$$

Plug this back into $(*)$, to finish:

$$F(z) = 2z^2 + z - 7.$$