The Fundamental Theorem of Calculus (FTC).

BIG IDEA: many definite integrals (like $\int_0^5 4x^2\,dx$) can be computed exactly (without Riemann sums).

Here's how: think of energy $E(t)$ and power $p(t)$. We've seen that:

Area under the graph of $p(t)$, from $t=a$ to $t=b$
= net energy (consumed/produced) from $t=a$ to $t=b$. (*)

Let's rewrite this, by noting that:

(1) The left side of (*) is just the integral $\int_a^b p(t)\,dt$.
(2) The right side of (*) is just:
final energy reading minus original energy reading
$= E(b) - E(a)$,
where $E(t)$ is our energy function.
(3) As we've seen,
$$E'(t) = p(t).$$

Putting all of this into (*) gives
$$\int_a^b p(t)\,dt = E(b) - E(a), \quad \text{where } E'(t) = p(t).$$

COOL FACT: all of this depends only on the relationship $E'(t) = p(t)$, NOT on the "real world" meanings of $E(t)$ and $p(t)$. Conclusion:

> If $F(x)$ is any function such that $F'(x) = f(x)$ on $[a,b]$, then
> $$\int_a^b f(x)\,dx = F(b) - F(a)$$

(FTC v. 1)

(In words: to integrate, antidifferentiate!)

Example 1. Find
$$\int_0^5 4x^2\,dx.$$

Solution.

We need to find an antiderivative of $4x^2$; meaning: find a function $F(x)$ with $F'(x)=4x^2$.

By "guessing and checking,"* we find that $F(x)=\frac{4}{3}x^3$ works, since
$$F'(x)=\frac{d}{dx}\left[\frac{4}{3}x^3\right]=\frac{4}{3}\cdot 3x^2=4x^2. \quad \checkmark$$

So, by FTC v. 1,
$$\int_0^5 4x^2\,dx = F(5)-F(0)=\frac{4}{3}\cdot 5^3-\frac{4}{3}\cdot 0^3$$
$$=\frac{4\cdot 125}{3}=\frac{500}{3}.$$

* Better strategies coming soon.

Intermission. Let's write $F(x)\Big|_a^b$ for $F(b)-F(a)$. Then FTC v. 1 reads

$$\boxed{\int_a^b f(x)\,dx = F(x)\Big|_a^b, \text{ where } F'(x)=f(x)\,dx.} \qquad \text{(FTC v. 2)}$$

More examples (compare with some earlier results):

2) $\int_0^5 4x\,dx = 2x^2\Big|_0^5 = 2\cdot 5^2-2\cdot 0^2 = 50.$

an antiderivative of $4x$ is $2x^2$, since $\frac{d}{dx}[2x^2]=4x$.

3) $\int_{-\pi}^{\pi} \sin(x)\,dx = -\cos(x)\Big|_{-\pi}^{\pi} = -\cos(\pi)-(-\cos(-\pi))$
$= -(-1)-(-(-1))=0.$

an antiderivative of $\cos(x)$ is $-\sin(x)$ (by guessing/checking)

4) $\int_{-2}^{4} x\,dx = \frac{x^2}{2}\Big|_{-2}^{4} = \frac{4^2}{2}-\frac{(-2)^2}{2} = \frac{16-4}{2}=6.$

an antiderivative of x is $x^2/2$.

5. $\int_1^6 \sqrt{x-1}\,dx = \frac{2}{3}(x-1)^{3/2}\Big|_1^6 = \frac{2}{3}\left((6-1)^{3/2}-(1-1)^{3/2}\right) = \frac{2}{3}\left(5^{3/2}-0^{3/2}\right)$

this antideriv. isn't obvious, but you can check it

$= \frac{2}{3}\sqrt{5^3}$

$= \frac{2}{3}\sqrt{125} = 7.45356\ldots$

6) $\int_0^{\pi/4} \cos(2x)\,dx = \frac{\sin(2x)}{2}\Big|_0^{\pi/4} = \frac{\sin(2\cdot\frac{\pi}{4})-\sin(2\cdot 0)}{2} = \frac{1-0}{2} = \frac{1}{2}.$

guess and check the antiderivative

7) $\int_0^1 e^{7y}\,dy = \frac{e^{7y}}{7}\Big|_0^1 = \frac{e^{7\cdot 1}-e^{7\cdot 0}}{7} = \frac{e^7-1}{7} = 156.51902\ldots$

8) $\underline{\int_{-2}^7 dx} = x\Big|_{-2}^7 = 7-(-2) = 9$

← notation: this means $\int_{-2}^7 1\,dx$
(integrate the constant function $f(x)=1$).

9) $\int_{-1}^1 \frac{dx}{1+x^2} = \int_{-1}^1 \frac{1}{1+x^2}\,dx = \arctan(x)\Big|_{-1}^1 = \arctan(1)-\arctan(-1)$

$= \pi/4 - (-\pi/4) = \frac{\pi}{2}$

(since $\tan(\pi/4)=1$ and $\tan(-\pi/4)=-1$).

10) $\int_{-2}^2 \frac{e^{-x^2/2}}{\sqrt{2\pi}}\,dx = ?$

There's no nice antiderivative of $f(x) = \frac{1}{\sqrt{2\pi}}$, so we can't use FTC. But we can use Riemann sums. E.g. using left endpoints and $n = 10{,}000$, we find that

$$\int_{-2}^2 \frac{e^{-x^2/2}}{\sqrt{2\pi}}\,dx \approx 0.9544973\ldots$$