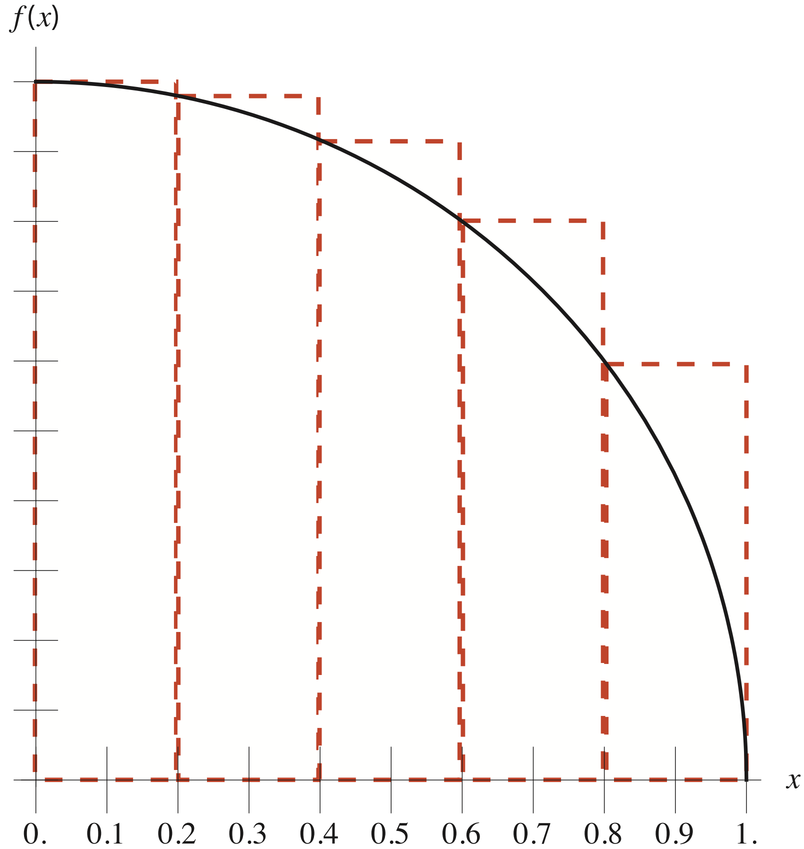
1. Below is a sketch of the function $f(x) = \sqrt{1 - x^2}$.



(a) On top of this sketch, draw in the rectangles that would represent a *left endpoint* Riemann sum approximation, with $n = 5$, to the definite integral $\int_0^1 f(x)\,dx$. See above.

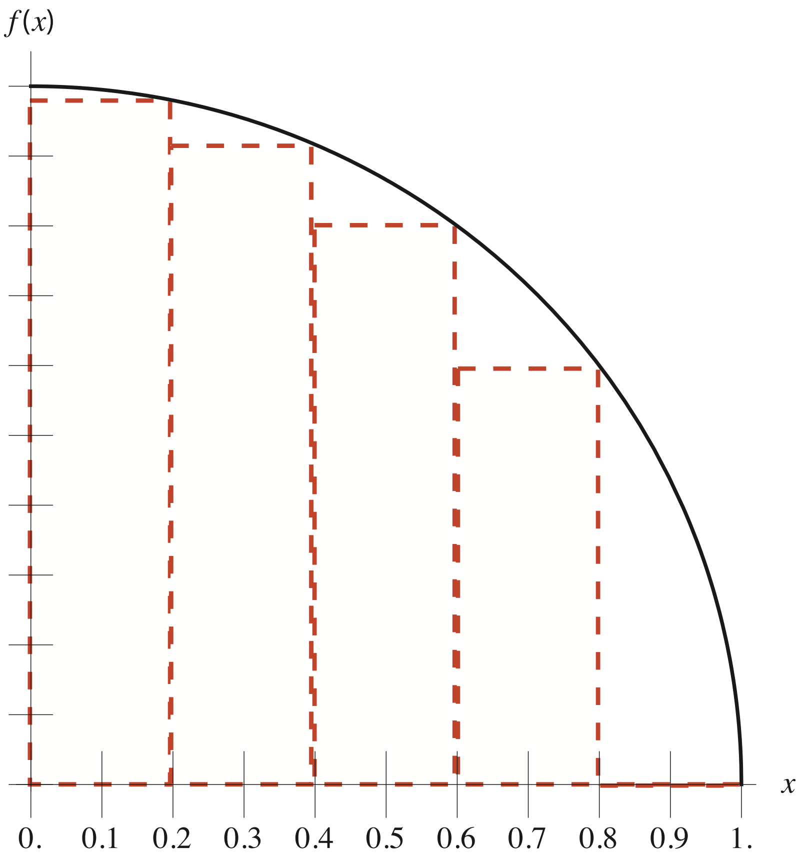
(b) Will your above left endpoint Riemann sum approximation, call it LEFT(5), be an overestimate or an underestimate of the above integral? Explain, without doing any computations (yet).

It will be an overestimate, because the rectangles overshoot the area under the curve.

(c) Use the Riemann sum represented in your above picture to approximate $\int_0^1 f(x)\,dx$. Give a numerical answer with at least four digits to the right of the decimal point.

$\int_0^1 f(x)\,dx \approx$ LEFT(5) = 0.8593.

2. Below, again, is a sketch of the function $f(x) = \sqrt{1 - x^2}$.



(a) On top of this sketch, draw in the rectangles that would represent a *right endpoint* Riemann sum approximation, with $n = 5$, to $\int_0^1 f(x)\,dx$. See above.

(b) Will your above right endpoint Riemann sum approximation, call it RIGHT(5), be an overestimate or an underestimate of the above integral? Explain, without doing any computations (yet).

It will be an underestimate, because the rectangles undershoot the area under the curve.

(c) Use the Riemann sum represented in your above picture to approximate $\int_0^1 f(x)\,dx$. Give a numerical answer with at least four digits to the right of the decimal point.

$\int_0^1 f(x)\,dx \approx$ RIGHT(5)= 0.6593.

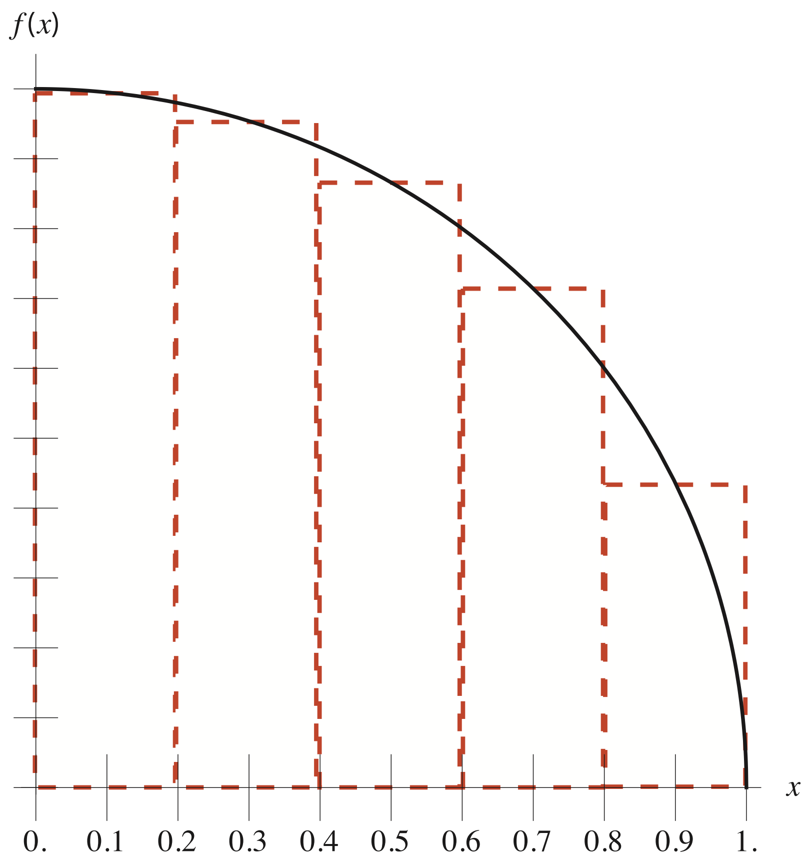
3. *Without actually computing* the quantities LEFT(2000) and RIGHT(2000), list the following in increasing order:

$$\text{LEFT}(2000),\ \text{RIGHT}(2000),\ \text{LEFT}(5),\ \text{RIGHT}(5),\ \int_0^1 f(x)\,dx$$

Please explain how you got your answer. All left endpoint approximations are overestimates; all right endpoint approximations are underestimates. Also, the larger the number of rectangles, in any given kind of approximation, the better the estimate (that is, the closer the approximation to the actual area). So

$$\text{RIGHT}(5) < \text{RIGHT}(2000) < \int_0^1 f(x)\,dx < \text{LEFT}(2000) < \text{LEFT}(5).$$

4. Below, yet *again*, is a sketch of the function $f(x) = \sqrt{1 - x^2}$.



(a) On top of this sketch, draw in the rectangles that would represent a *midpoint* Riemann sum approximation, with $n = 5$, to $\int_0^1 f(x)\,dx$. See above.

(b) Use the Riemann sum represented in your above picture to approximate $\int_0^1 f(x)\,dx$. Give a numerical answer with at least four digits to the right of the decimal point.
$\int_0^1 f(x)\,dx \approx$ MID(5)= 0.7930.

(c) Do you think your approximation from part (b) of this exercise is an underestimate or an overestimate? Please explain.

Each rectangle partially undershoots and partially overshoots the area under the curve. It looks like the approximation is pretty close to the true area. But it seems, especially in the rectangle on the far right, that the overshoot of each rectangle is just *slightly* greater, in area, than its undershoot. So we expect to get a slight overestimate.

5. What is $\int_0^1 f(x)\,dx$ *exactly* (to four decimal places)? Hint: again, an integral is an area. And as you can see, the graph of $f(x)$ on $[0, 1]$ describes a quarter circle. So you can use an appropriate area formula from high school geometry.

$$\int_0^1 f(x)\,dx = \frac{\pi}{4} \approx 0.7854.$$

Note that this confirms our answer to Exercise 4(c) above.