Week 12 - Tuesday, 11/10

Integrals: the story so far.

I. Summary.

Given a function $f(x)$, let $F(x)$ be an antiderivative for $f(x)$, meaning

$$F'(x) = f(x).$$

Then:

(A) The definite integral $\int_a^b f(x)\,dx$, which is the signed area between $f(x)$ and the interval $[a,b]$ on the x-axis, is given by FTC:

$$\int_a^b f(x)\,dx = F(x)\Big|_a^b = F(b) - F(a).$$

(B) The indefinite integral $\int f(x)\,dx$, which is the set of all antiderivatives of $f(x)$, is given by the COOL FACT of last time:

$$\int f(x)\,dx = F(x) + C$$

where C denotes an arbitrary constant.

II. Short table of indefinite integrals.

(Here, a, b, and p are constants.)

$f(x)$	$\int f(x)\,dx$
x^p	$\frac{x^{p+1}}{p+1} + C \quad (p \neq -1)$
$1/x$	$\ln(\lvert x \rvert) + C$
$\sin(ax)$	$\frac{-\cos(ax)}{a} + C \quad (a \neq 0)$
$\cos(ax)$	$\frac{\sin(ax)}{a} + C \quad (a \neq 0)$

(continued)

$f(x)$	$\int f(x)\,dx$
e^{ax}	$\frac{e^{ax}}{a} + C \quad (a \neq 0)$
b^x	$\frac{b^x}{\ln(b)} + C \quad (b > 0)$
$\frac{1}{1+x^2}$	$\arctan(x) + C$

Note that we can <u>check</u> the above table by differentiation:
e.g.

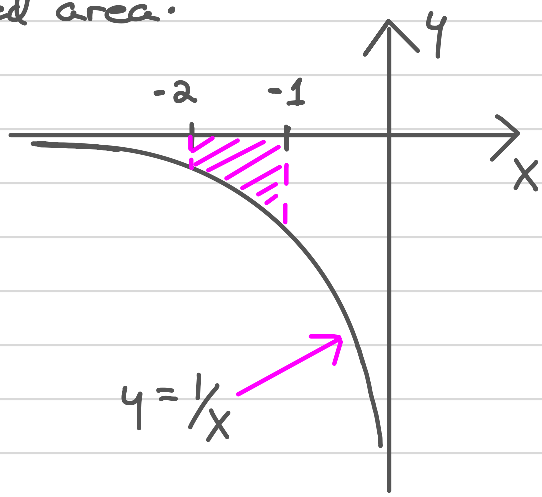
$$\frac{d}{dx}\left[\frac{\sin(ax)}{a}+C\right] = \frac{1}{a}\frac{d}{dx}[\sin(ax)] + \frac{d}{dx}[C]$$
$$= \frac{1}{a}\cdot\cos(ax)\cdot a + 0 = \cos(ax). \checkmark$$

III. Examples: use the above table to find the indicated quantities.

(1) $\displaystyle\int \left(\frac{3}{x^2} + 3x^2 - 3\sqrt{x} + \frac{3}{\sqrt{x}}\right) dx$

(2) $\displaystyle\int_0^{\pi/2} (\cos(2x) - 2\sin(x))\, dx$

(3) The signed area:



Solutions

(1) $$\int\left(\frac{3}{x^2}+3x^2-3\sqrt{x}+\frac{3}{\sqrt{x}}\right)dx=\int\left(3x^{-2}+3x^2-3x^{1/2}+3x^{-1/2}\right)dx$$

$$=\frac{3x^{-1}}{-1}+\frac{3x^3}{3}-\frac{3x^{3/2}}{3/2}+\frac{3x^{1/2}}{1/2}+C=-3x^{-1}+x^3-2x^{3/2}+6x^{1/2}+C.$$

(2) $$\int_0^{\pi/2}(\cos(2x)-2\sin(x))\,dx=\left(\frac{\sin(2x)}{2}+2\cos(x)\right)\Big|_0^{\pi/2}$$

$$=\left(\frac{\sin(2\cdot\pi/2)}{2}+2\cos(\pi/2)\right)-\left(\frac{\sin(2\cdot 0)}{2}+2\cos(0)\right)$$

$$=(0/2+2\cdot 0)-(0/2+2)=-2.$$

(3) The signed area is

$$\int_{-2}^{-1}\frac{1}{x}\,dx=\ln(|x|)\Big|_{-2}^{-1}=\ln(|-1|)-\ln(|-2|)$$

$$=\ln(1)-\ln(2)=0-\ln(2)=-\ln(2)\approx-0.6931.$$

Note: without the absolute values, we'd be trying to evaluate things like $\ln(-2)$, which doesn't make sense.