Antiderivatives

Recall the Fundamental Theorem of Calculus (FTC):

$$\int_a^b f(x)\,dx = F(x)\Big|_a^b = F(b) - F(a),$$

where $F(x)$ is any antiderivative of $f(x)$, meaning $F'(x) = f(x)$.

Example 1.
(a) Find two antiderivatives $F(x)$ and $G(x)$ of $f(x) = x^2$.
(b) Evaluate $\int_1^4 x^2\,dx$ in two ways.

Solution.
(a) $F(x) = \frac{x^3}{3}$ works, since $F'(x) = \frac{1}{3}\cdot 3x^2 = x^2 = f(x)$. But so does
$G(x) = \frac{x^3}{3} + 5$, since $G'(x) = \frac{1}{3}\cdot 3x^2 + 0 = x^2 = f(x)$.

(b) By FTC,

$$\int_1^4 x^2\,dx = \frac{x^3}{3}\Big|_1^4 = \frac{4^3}{3} - \frac{1^3}{3} = \frac{64-1}{3} = \frac{63}{3} = 21.$$

OR

$$\int_1^4 x^2\,dx = \left(\frac{x^3}{3} + 5\right)\Big|_1^4 = \left(\frac{4^3}{3} + \cancel{5}\right) - \left(\frac{1^3}{3} + \cancel{5}\right)$$

$$= \frac{4^3}{3} - \frac{1^3}{3} = \frac{64-1}{3} = \frac{63}{3} = 21.$$

COOL FACT:

in general, if $F(x)$ is one antiderivative of $f(x)$, then any antiderivative of $f(x)$ is of the form

$$F(x)+C$$

for some constant C.

Let's write

$$\int f(x)\,dx$$

"the indefinite integral of $f(x)\,dx$"

for the set of all antiderivatives of $f(x)$.

Examples: find these indefinite integrals and check your work.

2) $\int \cos(x)\,dx = \sin(x)+C.$

The COOL FACT says: to find $\int f(x)\,dx$,

a) first find some (any) antiderivative of $f(x)$;

b) then tack on the arbitrary constant "$+C$."

DON'T FORGET THIS STEP!!

Check:

$$\frac{d}{dx}[\sin(x)+C] = \cos(x)+0 = \cos(x) \checkmark$$

3) $\int \sin(x)\,dx = -\cos(x)+C.$

Check:

$$\frac{d}{dx}[-\cos(x)+C] = -(-\sin(x))+0 = \sin(x).$$

4) $\int \sin(2x)\,dx = \dfrac{-\cos(2x)}{2} + C.$

Check:
$$\frac{d}{dx}\left[\frac{-\cos(2x)}{2} + C\right] = \frac{-1}{2}(-\sin(2x)\cdot 2) + 0 = \sin(2x). \checkmark$$

5) $\int e^{7y}\,dy = \dfrac{e^{7y}}{7} + C.$

Check: $\dfrac{d}{dy}\left[\dfrac{e^{7y}}{7} + C\right] = \dfrac{1}{7}\cdot e^{7y}\cdot 7 + 0 = e^{7y}. \checkmark$

6) For any $b>0$,
$$\int b^x\,dx = \frac{b^x}{\ln(b)} + C.$$
Check: $\dfrac{d}{dx}\left[\dfrac{b^x}{\ln(b)} + C\right] = \dfrac{1}{\ln(b)}\cdot \ln(b)\, b^x + 0 = b^x. \checkmark$

7) $\int (3t^2 - 7t + 2)\,dt = t^3 - \frac{7}{2}t^2 + 2t + C.$

Check: DIY.

Remarks on definite vs. indefinite integrals.

A) The FTC says: for definite integrals, proceed essentially as above, but replace the "$+C$" with "$\Big|_a^b$".

For example:

8) $\displaystyle\int_0^{\pi/4} \sin(2x)\,dx = \frac{-\cos(2x)}{2}\Big|_0^{\pi/4} = \frac{-\cos(2\cdot \pi/4)}{2} - \left(\frac{-\cos(2\cdot 0)}{2}\right) = \frac{-0}{2} + \frac{1}{2} = \frac{1}{2}.$

9) $\displaystyle\int_0^1 e^{7y}\,dy = \frac{e^{7y}}{7}\Big|_0^1 = \frac{e^{7\cdot 1}}{7} - \frac{e^{7\cdot 0}}{7} = \frac{e^7 - e^0}{7} = \frac{e^7 - 1}{7}.$

B) So: a definite integral is a number; an indefinite integral is a whole set of functions.

For example:

10) $\int x^4\,dx = \dfrac{x^5}{5} + C$

while

$$\int_1^3 x^4\,dx = \left.\frac{x^5}{5}\right|_1^3 = \frac{3^5}{5} - \frac{1^5}{5} = \frac{243-1}{5} = \frac{242}{5}.$$