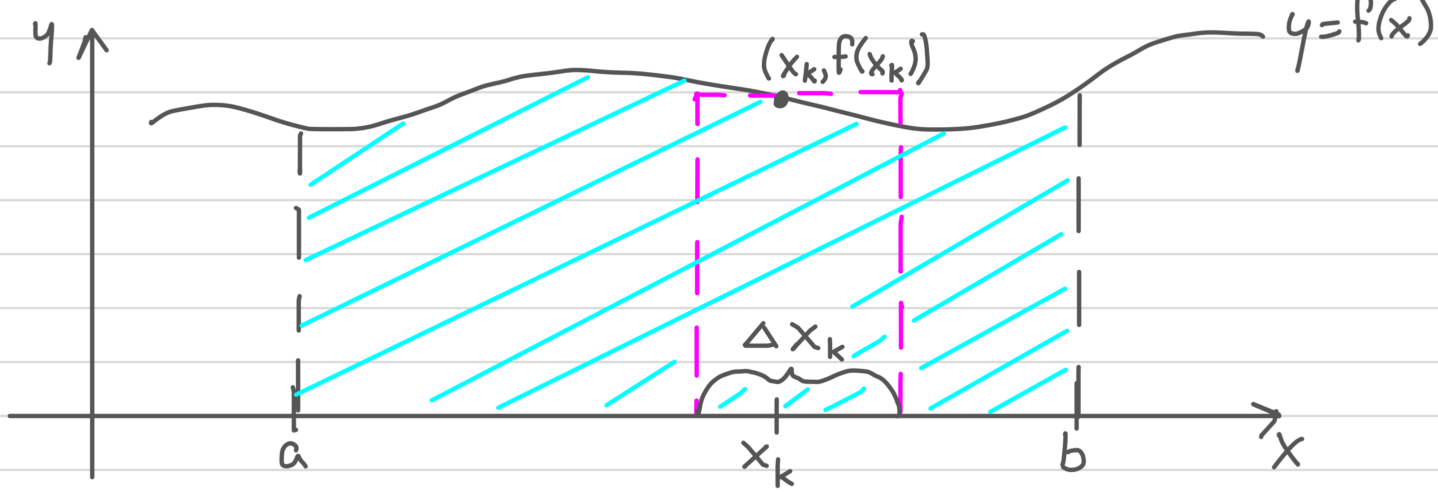
Week 11 - Wednesday, 11/4

The definite integral.

Let $f(x)$ be a function that's ≥ 0 on $[a,b]$.



The above shaded area is denoted

$$\int_a^b f(x)\,dx :$$

"the definite integral, from a to b, of $f(x)\,dx$."

Note: for now, the "dx" is just a placeholder.

Recall that this area can be approximated by Riemann sums, which give better approximations as we use more, and narrower, rectangles. SO:

(S)
$$\boxed{\int_a^b f(x)\,dx = \lim_{\substack{\text{all } \Delta x_k\text{'s} \\ \to 0}} \left(f(x_1)\Delta x_1 + f(x_2)\Delta x_2 + \dots + f(x_n)\Delta x_n \right).}$$

← Definition of $\int_a^b f(x)\,dx$

As usual, x_k is a point in the k^{th} subinterval, and Δx_k is the length of that subinterval.

Since $\int_a^b f(x)\,dx$ is an area (for $f(x) \geq 0$ on $[a,b]$), we can **sometimes** evaluate definite integrals by basic geometry.

Examples (DIY: draw a picture, if it helps).

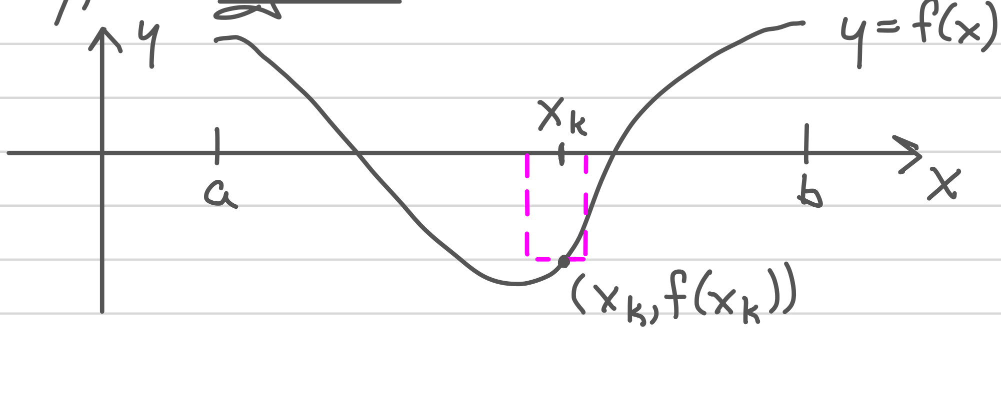
1) $\int_0^5 4\,dx$ = area under the graph of $f(x)=4$, over $[0,5]$ = base · height
$= 5 \cdot 4 = 20.$

2) $\int_0^5 4x\,dx$ = area under the graph of $f(x)=4x$, over $[0,5]$ = ½ · base · height
$= \frac{1}{2} \cdot 5 \cdot 20 = 50.$

3) $\int_0^5 4x^2\,dx$ = area under the graph of $f(x)=4x^2$, over $[0,5]$ = ?

We could approximate this area with Riemann sums, but soon, we'll see a better way!

// Remark. If $f(x) < 0$ somewhere on $[a,b]$, then some of the summands $f(x_k)\Delta x_k$ on the right side of (S) will, typically, be negative.



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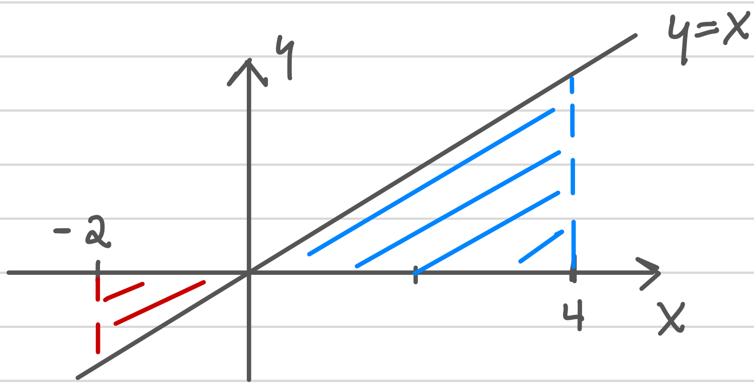
Consequence (DIY: think about it): in general (even if $f(x)<0$ in places),

($\int'$) $$\boxed{\begin{aligned}\int_a^b f(x)\,dx &= \underline{\text{signed}} \text{ area of } f(x) \text{ from } a \text{ to } b \\ &= \text{sum of areas above x-axis, MINUS sum of areas below}\end{aligned}}$$

Examples.

4) $\int_{-2}^{4} x\,dx =$



BLUE AREA minus
RED AREA

$= \frac{1}{2}\cdot 4\cdot 4 - \frac{1}{2}\cdot 2\cdot 2 = 8-2=6.$

5) $\int_{-\pi}^{\pi} \sin(x) = 0,$

since the areas of the two "lobes" cancel.