Week 11 - Tuesday, 11/3

Digression: Let's have "sum" fun.

A) We consider the sum of the first however-many positive integers. That is, we look at sums like

$$1+2+3+\dots+12$$
$$1+2+3+\dots+199{,}999$$
$$1+2+3+\dots+n \qquad (*).$$

Let's call the last sum S. To compute it, we'll write it down forwards and backwards, and then add in columns. Like this:

$$\begin{array}{rl} S = & 1 + 2 + 3 + 4 + 5 + \dots + (n-1) + n. \\ S = & n + (n-1) + (n-2) + (n-3) + (n-4) + \dots + 2 + 1. \\ \hline \text{Add: } 2S = & (n+1) + (n+1) + (n+1) + (n+1) + (n+1) + \dots + (n+1) + (n+1). \end{array}$$

Each column of numbers on the right sums to $n+1$. There are n columns, so we get

$$2S = n(n+1). \qquad \text{Solve for } S:$$

$$\boxed{S = \frac{n(n+1)}{2}}$$ formula for the sum of the first n whole numbers

E.g.

$$1+2+3+\dots+12 = \frac{12(12+1)}{2} = 78,$$

$$1+2+3+\dots+199{,}999 = \frac{199{,}999(200{,}000)}{2}$$
$$= 19{,}999{,}900{,}000 \text{, etc.}$$

B) Application to Riemann sums.

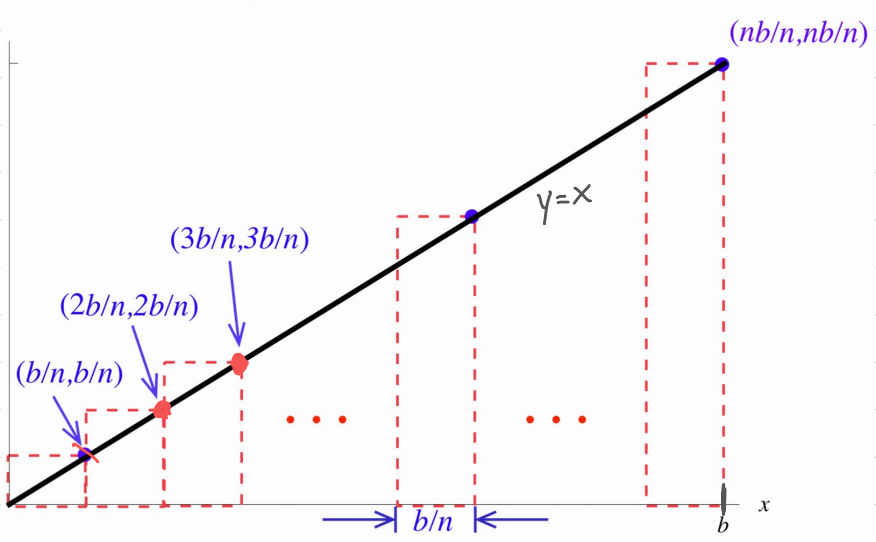
Exercise:

(1) Use a right endpoint Riemann sum, with n evenly spaced rectangles, to approximate the area A under the graph of $y=x$, from $x=0$ to $x=b$.

(2) Find A exactly.

Solution:

(a) Here's a picture:



We have:

RIGHT(n) = sum of areas of above rectangles

$$= \frac{b}{n}\cdot\frac{b}{n} + \frac{b}{n}\cdot\frac{2b}{n} + \frac{b}{n}\cdot\frac{3b}{n} + \dots + \frac{b}{n}\cdot\frac{nb}{n}$$

$$= \frac{b}{n}\cdot\frac{b}{n}(1+2+3+\dots+n)$$

by part (A) above

$$= \frac{b^2}{n^2}\cdot\frac{n(n+1)}{2} = \frac{b^2}{2}\cdot\frac{n^2+n}{n^2} = \frac{b^2}{2}\left(1+\frac{1}{n}\right).$$

(b) To get the exact area, we let $n\to\infty$. We get

$$A = \lim_{n\to\infty}\frac{b^2}{2}\left(1+\frac{1}{n}\right) = \frac{b^2}{2},$$

since $1/n \to 0$ as $n\to\infty$.

C) Just some wacky stuff about infinite sums.

In part (A) above, we computed the sum of the first n positive integers. Now we evaluate the sum of <u>all</u> positive integers (and get a surprise)!

1) We first need to find

$$1+1-1+1-1+1-1+\dots \quad \text{(ad infinitum)}.$$

To do this, let's call the sum A:

$$
\begin{array}{lrl}
 & A = & 1 - 1 + 1 - 1 + 1 - 1 + \dots \qquad \text{Write it again:} \\
 & A = & \quad\;\; 1 - 1 + 1 - 1 + 1 - \dots \\
\hline
\text{Add:} & 2A = & 1 + 0 + 0 + 0 + 0 + 0 + \dots
\end{array}
$$

The first column on the right sums to 1; all others sum to zero. So we get $2A = 1$, or

$$\boxed{A = 1/2.}$$

2) We'll also need to find

$$1 - 2 + 3 - 4 + 5 - 6 + 7 - 8 + \ldots \qquad \text{(ad infinitum).}$$

To do so, let's call this sum B:

$$B = 1 - 2 + 3 - 4 + 5 - 6 + 7 - \ldots \quad \text{Write it again:}$$
$$B = \quad\;\; 1 - 2 + 3 - 4 + 5 - 6 + \ldots$$

Add: $2B = 1 - 1 + 1 - 1 + 1 - 1 + 1 - \ldots$

The right-hand side is what we called A, above. So $2B = A$, or

$$\boxed{B = \frac{A}{2} = \frac{1/2}{2} = \frac{1}{4}}$$

since, as we saw above, $A = 1/2$.

3) Finally, we evaluate the sum of all positive integers, call it C. That is,

$$C = 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 - \ldots \quad \text{Also, recall:}$$
$$B = 1 - 2 + 3 - 4 + 5 - 6 + 7 - 8 + \ldots \quad \text{Now subtract:}$$
$$C - B = 0 + 4 + 0 + 8 + 0 + 12 + 0 + 16 + \ldots$$
$$= 4 + 8 + 12 + 16 + \ldots$$

$$= 4(1+2+3+4+\dots)$$
$$= 4C.$$

So $C - B = 4C$. Solve for C:

$$-3C = B$$

$$\boxed{C = \frac{B}{-3} = \frac{1/4}{-3} = -\frac{1}{12}}$$

since, as we've seen, $B = 1/4$.

To repeat:

$$1+2+3+4+5+\dots = -\frac{1}{12}:$$

the sum of all positive integers is neither an integer nor positive !!