Week 11 - Thursday, 11/5

The Fundamental Theorem of Calculus (FTC).

BIG IDEA: many definite integrals (like $\int_0^5 4x^2\,dx$) can be computed exactly (without Riemann sums), using:

> If $F(x)$ is any function such that $F'(x) = f(x)$ on $[a,b]$, then
> $$\int_a^b f(x)\,dx = F(b) - F(a)$$

(FTC v. 1)

In words: to integrate, first antidifferentiate, meaning: find a function F whose derivative is the function f being considered.

Examples. (DIY: compare examples (1)–(4) here with some examples from last time.)

(1) Find
$$\int_0^5 4x^2\,dx.$$

Solution.

We need to find an antiderivative of $4x^2$, meaning: find a function $F(x)$ with $F'(x) = 4x^2$.

By "guessing and checking," we find that $F(x) = \frac{4}{3}x^3$ works, since
$$F'(x) = \frac{d}{dx}\left[\frac{4}{3}x^3\right] = \frac{4}{3}\cdot 3x^2 = 4x^2. \quad \checkmark$$

So, by FTC v. 1,
$$\int_0^5 4x^2\,dx = F(5) - F(0) = \frac{4}{3}\cdot 5^3 - \frac{4}{3}\cdot 0^3$$
$$= \frac{4\cdot 125}{3} = \frac{500}{3}.$$

Intermission. Let's write $F(x)\Big|_a^b$ for $F(b)-F(a)$. Then FTC v. 1 reads

$$\boxed{\int_a^b f(x)\,dx = F(x)\Big|_a^b, \text{ where } F'(x)=f(x).} \qquad \text{(FTC v. 2)}$$

More examples.

2) $\int_0^5 4x\,dx = 2x^2\Big|_0^5 = 2\cdot 5^2 - 2\cdot 0^2 = 50.$

an antiderivative of $4x$ is $2x^2$, since $\frac{d}{dx}[2x^2]=4x$.

3) $\int_{-\pi}^{\pi} \sin(x)\,dx = -\cos(x)\Big|_{-\pi}^{\pi} = -\cos(\pi) - (-\cos(-\pi))$
$= -(-1) - (-(-1)) = 0.$

an antiderivative of $\sin(x)$ is $-\cos(x)$ (by guessing/checking)

4) $\int_{-2}^{4} x\,dx = \frac{x^2}{2}\Big|_{-2}^{4} = \frac{4^2}{2} - \frac{(-2)^2}{2} = \frac{16-4}{2} = 6.$

an antiderivative of x is $x^2/2$.

5. $\int_1^6 \sqrt{x-1}\,dx = \frac{2}{3}(x-1)^{3/2}\Big|_1^6 = \frac{2}{3}\left((6-1)^{3/2} - (1-1)^{3/2}\right) = \frac{2}{3}(5^{3/2} - 0^{3/2})$

this antideriv. isn't obvious, but you can check it

$= \frac{2}{3}\sqrt{5^3}$

$= \frac{2}{3}\sqrt{125} = 7.45356\ldots$

6) $\int_0^{\pi/4} \cos(2x)\,dx = \frac{\sin(2x)}{2}\Big|_0^{\pi/4} = \frac{\sin(2\cdot\pi/4) - \sin(2\cdot 0)}{2} = \frac{1-0}{2} = \frac{1}{2}.$

guess and check the antiderivative

7) $\int_0^1 e^{7y}\,dy = \frac{e^{7y}}{7}\Big|_0^1 = \frac{e^{7\cdot 1} - e^{7\cdot 0}}{7} = \frac{e^7 - 1}{7} = 156.51902\ldots$

8) $\underbrace{\int_{-2}^{7} dx}$ $= x\Big|_{-2}^{7} = 7-(-2) = 9$

← notation: this means $\int_{-2}^{7} 1\, dx$
(integrate the constant function $f(x)=1$).

9) $\displaystyle\int_{-1}^{1} \frac{dx}{1+x^2} = \int_{-1}^{1} \frac{1}{1+x^2}\, dx = \arctan(x)\Big|_{-1}^{1} = \arctan(1) - \arctan(-1)$

$= \pi/4 - (-\pi/4) = \frac{\pi}{2}$

(since $\tan(\pi/4) = 1$ and $\tan(-\pi/4) = -1$).

10) $\displaystyle\int_{-2}^{2} \frac{e^{-x^2/2}}{\sqrt{2\pi}}\, dx = ?$

There's no nice antiderivative of $f(x) = \frac{1}{\sqrt{2\pi}}$, so we can't use FTC. But we can use Riemann sums. E.g. using midpoints and $n = 10{,}000$, we find that

$$\int_{-2}^{2} \frac{e^{-x^2/2}}{\sqrt{2\pi}}\, dx \approx 0.95450.$$

Next time: a "proof" of FTC.