Week 11 - Friday, 11/6

Digression: what if dx were one of us?
Alternative title: calculus, unlimited.

Imagine a (fantastical) number dx that's:

- the smallest possible positive number;
- nonzero when we need it to be;
- zero when we want it to be.

Then all sorts of calculus can happen without needing limits. For example:

1) Derivatives.

Given a function $f(x)$, define the "infinitesimal net change in $F(x)$, from x to $x+dx$," denoted dF, by

$$dF = \text{new value of } F \text{ minus old value} = F(x+dx) - F(x).$$

Then define the derivative of F with respect to x to be dF divided by the corresponding change in x: that is, define this derivative to be the fraction (of infinitesimals)

$$\frac{dF}{dx} = \frac{F(x+dx) - F(x)}{dx}.$$

(No " $\lim_{\Delta x \to 0}$ " required!)

Using this, we can do things like compute derivatives, without using limits.

Example: find dF/dx if $F(x) = x^2$.

Solution:

$$\frac{dF}{dx} = \frac{F(x+dx) - F(x)}{dx}$$

for the moment, we need $dx \neq 0$, so we can divide by dx.

$$= \frac{(x+dx)^2 - x^2}{dx}$$

$$= \frac{\cancel{x^2} + 2x\,dx + (dx)^2 - \cancel{x^2}}{dx}$$

$$= \frac{2x\,dx + (dx)^2}{dx} = \frac{\cancel{dx}(2x+dx)}{\cancel{dx}}$$

$$= 2x + dx = 2x.$$

now, we want $dx = 0$, to simplify things

2) Proof of the Fundamental Theorem of Calculus.

We want to show that, if

$$\frac{dF}{dx} = f(x) \qquad (*)$$

then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$ Here's how:

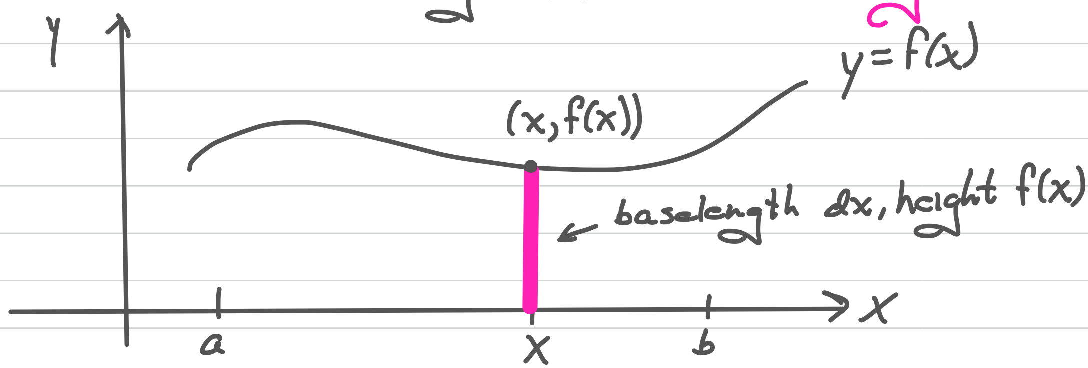
Assume (*) is true. We look at the net change ΔF, over an interval $[a,b]$, in two ways:

A) On the one hand, in the usual way:

ΔF on $[a,b]$ = new value − old value
= $F(b) - F(a)$.

B) On the other hand, let's break $[a,b]$ up into infinitely many intervals, each of length dx. Then

ΔF on $[a,b]$ = sum of changes dF in F over each of these intervals
= sum of quantities $f(x)\,dx$ over all x in $[a,b]$ — by (*)
= sum of (possibly, signed) areas of rectangles like the one in magenta:



But adding up (possibly, signed) areas of rectangles like the one above, from $x=a$ to $x=b$, clearly gives the (possibly, signed) area

$$\int_a^b f(x)\,dx.$$

So ΔF on $[a,b] = \int_a^b f(x)\,dx$.

Comparing (A) and (B) above gives

$$\int_a^b f(x)\,dx = F(b) - F(a),$$

and we're done.

ATWMR*

* "And there was much rejoicing."