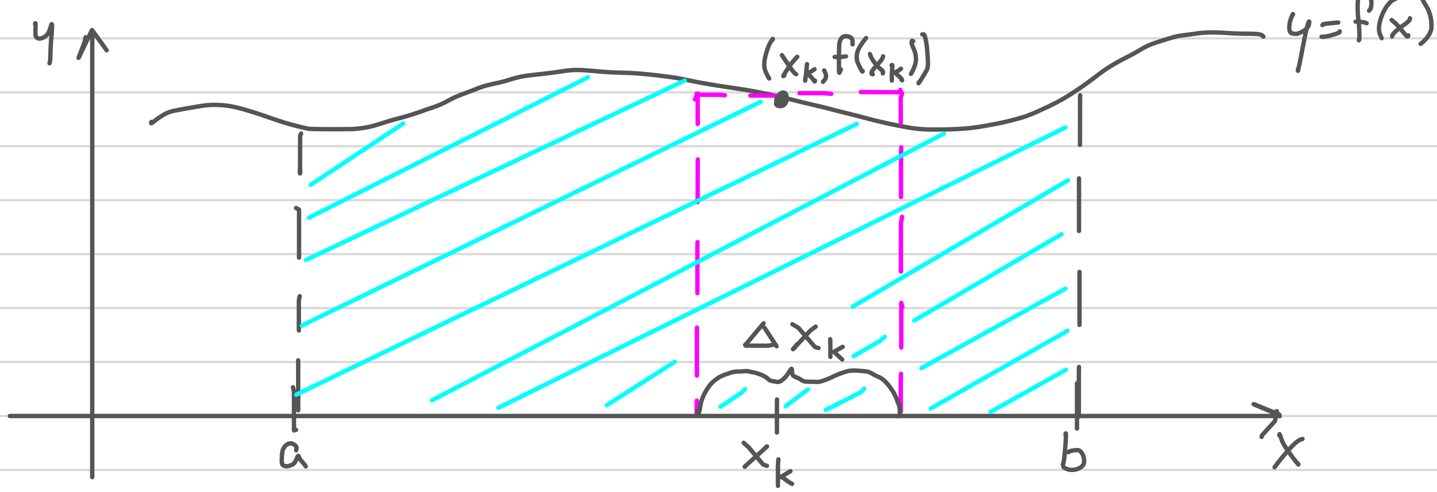
Week 10 - Wednesday, 3/20

## The definite integral.

Let $f(x)$ be a function that's $\geq 0$ on $[a,b]$.




The above shaded area is denoted

$$\int_a^b f(x)\,dx :$$

"the definite integral, from a to b, of $f(x)\,dx$."

Note: for now, the "$dx$" is just a placeholder.

Recall that this area can be approximated by Riemann sums, which give better approximations as we use more, and narrower, rectangles. SO:

(S)

$$\boxed{\int_a^b f(x)\,dx = \lim_{\substack{\text{all } \Delta x_k\text{'s} \\ \to 0}} \left( f(x_1)\Delta x_1 + f(x_2)\Delta x_2 + \ldots + f(x_n)\Delta x_n \right).}$$

↖ Definition of $\int_a^b f(x)\,dx$

As usual, $x_k$ is a point in the $k^{th}$ subinterval, and $\Delta x_k$ is the length of that subinterval.

Since $\int_a^b f(x)\,dx$ is an area (for $f(x) \geq 0$ on $[a,b]$), we can sometimes evaluate definite integrals by basic geometry.

Examples (DIY: draw a picture, if it helps).

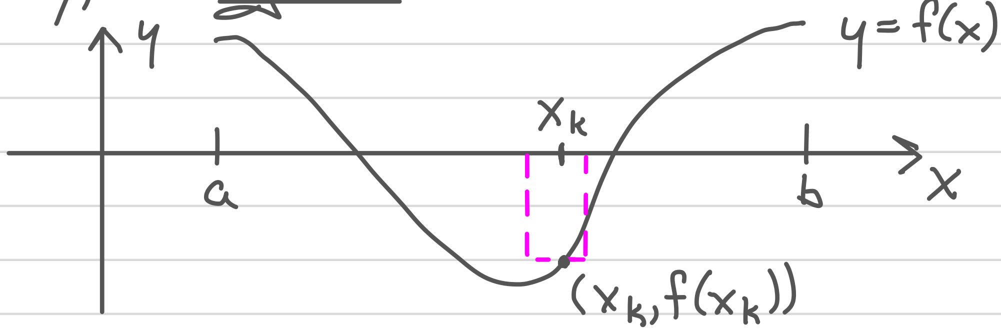
1) $\int_0^5 4\,dx$ = area under the graph of $f(x)=4$, over $[0,5]$ = base · height
$= 5 \cdot 4 = 20.$

2) $\int_0^5 4x\,dx$ = area under the graph of $f(x)=4x$, over $[0,5]$ = ½ · base · height
$= \frac{1}{2} \cdot 5 \cdot 20 = 50.$

3) $\int_0^5 4x^2\,dx$ = area under the graph of $f(x)=4x^2$, over $[0,5]$ = ?

We could approximate this area with Riemann sums, but soon, we'll see a better way!

// Remark. If $f(x) < 0$ somewhere on $[a,b]$, then some of the summands $f(x_k)\Delta x_k$ on the right side of (S) will, typically, be negative.

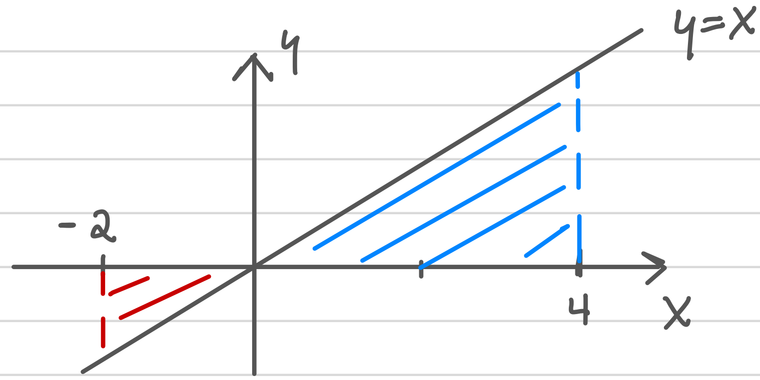


Consequence (DIY: think about it): in general (even if $f(x)<0$ in places),

($\int'$) $\boxed{\begin{aligned} \int_a^b f(x)\,dx &= \underline{\text{signed}} \text{ area of } f(x) \text{ from } a \text{ to } b \\ &= \text{sum of areas above x-axis, MINUS sum of areas below} \end{aligned}}$

<u>Examples.</u>

4) $\int_{-2}^{4} x\,dx =$

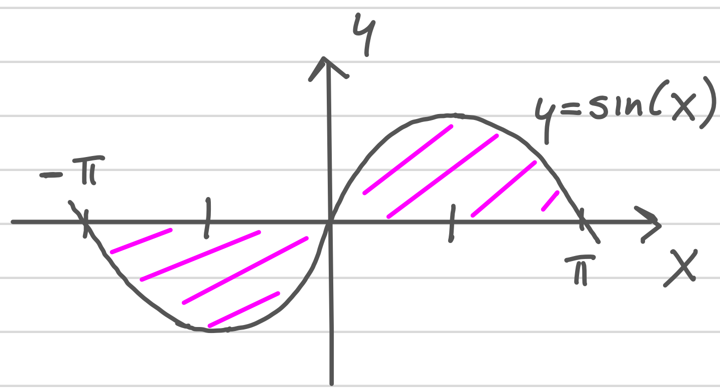



BLUE AREA minus
RED AREA

$= \frac{1}{2}\cdot 4\cdot 4 - \frac{1}{2}\cdot 2\cdot 2 = 8-2=6.$

5) $\int_{-\pi}^{\pi} \sin(x) = 0,$

since the areas of the two "lobes" cancel.




Final note: <u>rules</u> for integrals.
We have:

(i) $\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$ for any constant $c$;

(ii) $\int_a^b (f(x)+g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$;

(iii) $\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$;

(iv) $\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$.

just a convention, but a useful one, as we'll see (e.g. when we do integration by substitution).

Example 6.

$$\int_{\pi}^{-\pi} (3-7\sin(x))\,dx = -\int_{-\pi}^{\pi} (3-7\sin(x))\,dx$$

$$= -\int_{-\pi}^{\pi} 3\,dx + 7\int_{-\pi}^{\pi} \sin(x)\,dx$$

$$= -(3\cdot 2\pi) + 7\cdot 0 = -6\pi,$$

by Rules (i), (ii), (iv), Example 5, and (the idea of) Example 1.