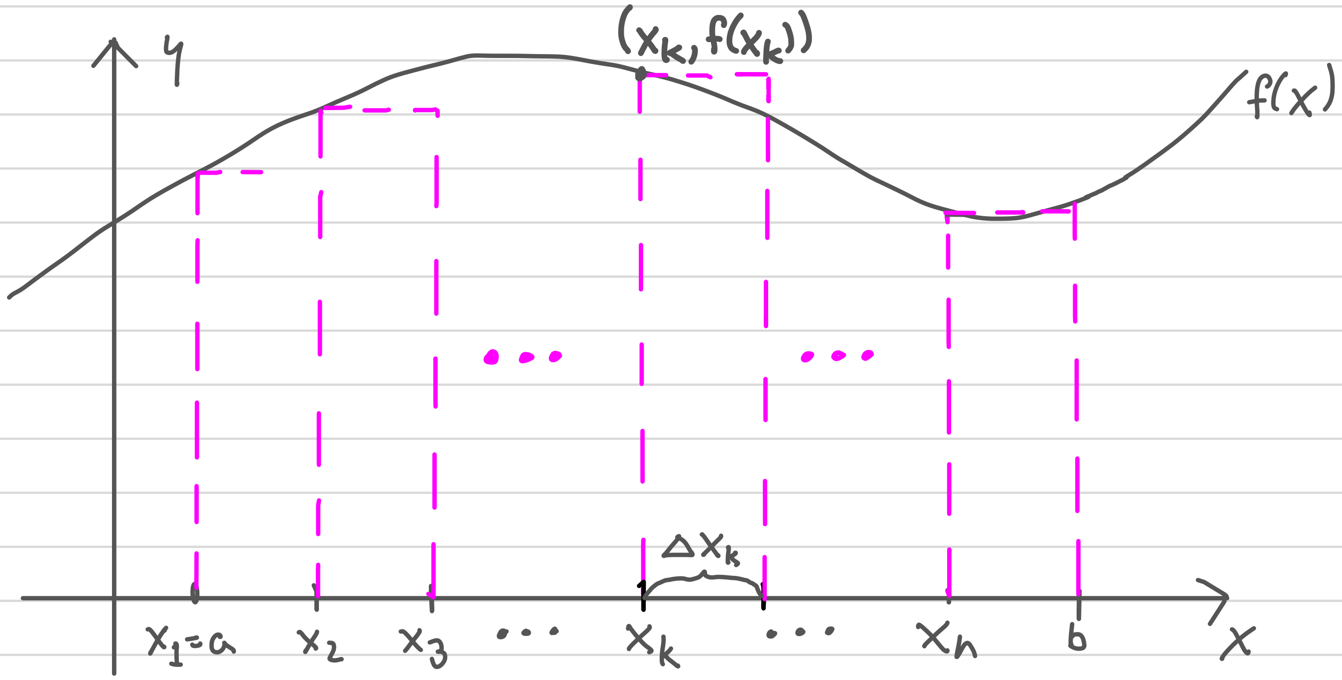
Riemann Sums.

Suppose we want to approximate the area under the graph of a function $y = f(x)$, over the interval $[a,b]$.



We break $[a,b]$ up into n subintervals, of lengths $\Delta x_1, \Delta x_2, \ldots, \Delta x_n$. We then pick one point in each subinterval. Call the k^{th} such point x_k.

Then

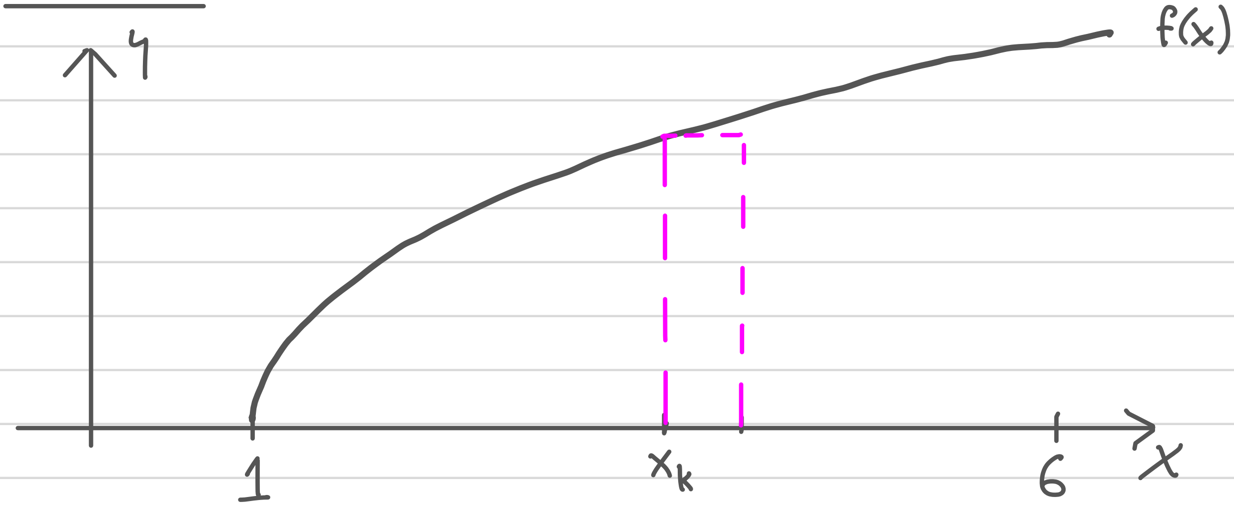
$$\text{(RS)} \qquad A \approx \underbrace{f(x_1)\Delta x_1 + f(x_2)\Delta x_2 + \ldots + f(x_n)\Delta x_n}_{\text{called a \underline{Riemann} \underline{sum} for } f \text{ on } [a,b]}.$$

(In the above picture, all Δx_k's are the same, and x_k = the left endpoint of the k^{th} subinterval, for $1 \le k \le n$.)

Example 1.

Approximate the area under the graph of $f(x) = \sqrt{x-1}$, over $[1,6]$, using 10 intervals of equal length, and left endpoints.

Solution.



Note that $\Delta x_k = \frac{6-1}{10} = 1/2$ for all k.
So by (RS),

$$A \approx f(1)\cdot\tfrac{1}{2} + f(3/2)\cdot\tfrac{1}{2} + f(2)\cdot\tfrac{1}{2} + \dots + f(11/2)\cdot\tfrac{1}{2}$$
$$= \tfrac{1}{2}\cdot\left(f(1) + f(3/2) + f(2) + \dots + f(11/2)\right)$$
$$= \tfrac{1}{2}\cdot\left(\sqrt{1-1} + \sqrt{3/2-1} + \sqrt{2-1} + \dots + \sqrt{11/2-1}\right)$$
$$= 6.8257.$$

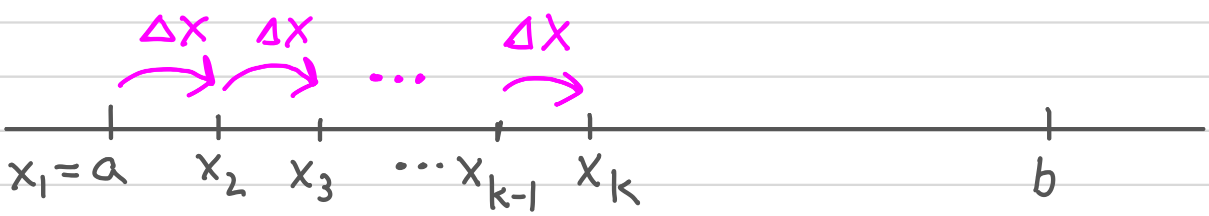
[Later, we'll see that the true area is
$$A = \frac{2}{3}\sqrt{125} = 7.4536\dots \;]$$

General notes.

(1) Until further notice, we'll take all subintervals to have equal length, call it Δx. Then, if we're using n subintervals, we have

$$\boxed{\Delta x = \frac{b-a}{n}.}$$

2a) A left endpoint Riemann sum is when x_k = the left endpoint of the k^{th} subinterval, for $1 \le k \le n$.



In this case,

$$x_k = a + (k-1)\Delta x$$

(start at $x=a$, then do $k-1$ jumps of length Δx), or by note 1,

$$\boxed{x_k = a + (k-1)\left(\frac{b-a}{n}\right).}$$

Formula for left endpoints x_k.

2b) Similarly, a right endpoint Riemann sum is when x_k = the right endpoint of the k^{th} subinterval for each k, so that

$$\boxed{x_k = a + k\left(\frac{b-a}{n}\right).}$$

Formula for right endpoints x_k.

(We jump k times instead of $k-1$.)

[There are also midpoint Riemann sums, where

$$x_k = a + (k - \tfrac{1}{2})\left(\frac{b-a}{n}\right),$$

and so on.]

Example 2.

In Example 1 above, what are the formulas for the left and right endpoints x_k?

Solution.

Left endpoints:

$$x_k = a + (k-1)\frac{(6-1)}{10} = 1 + (k-1)\cdot\frac{5}{10} = 1 + \frac{1}{2}(k-1) \quad (1 \le k \le 10).$$

Right endpoints:

$$x_k = a + k\cdot\frac{(6-1)}{10} = 1 + k\cdot\frac{5}{10} = 1 + \frac{1}{2}k \quad (1 \le k \le 10).$$