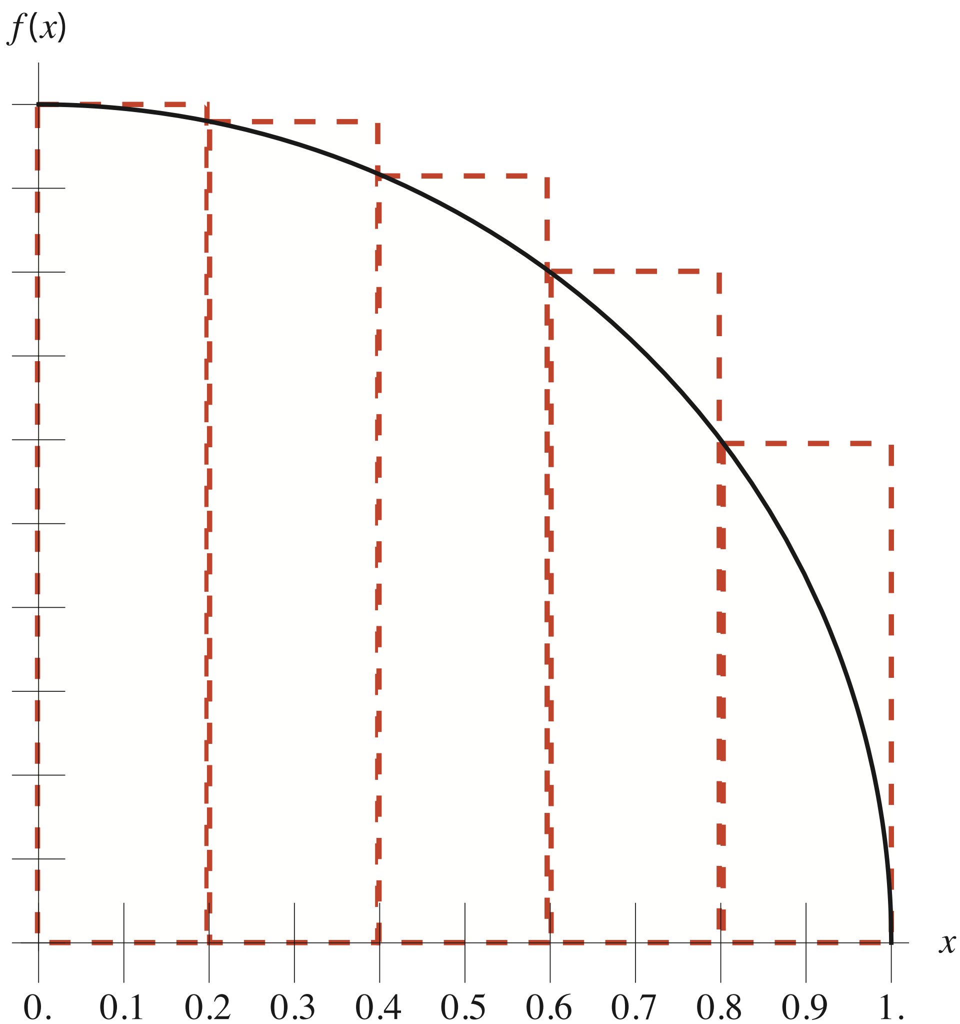
1. Below is a graph of the function $f(x) = \sqrt{1 - x^2}$.



Let's denote by the area under the graph of $f(x)$, over the interval $[0, 1]$, by A.

(a) On top of this graph, draw in the rectangles that would represent a *left endpoint* Riemann sum approximation, with $n = 5$, to the area A. See above.

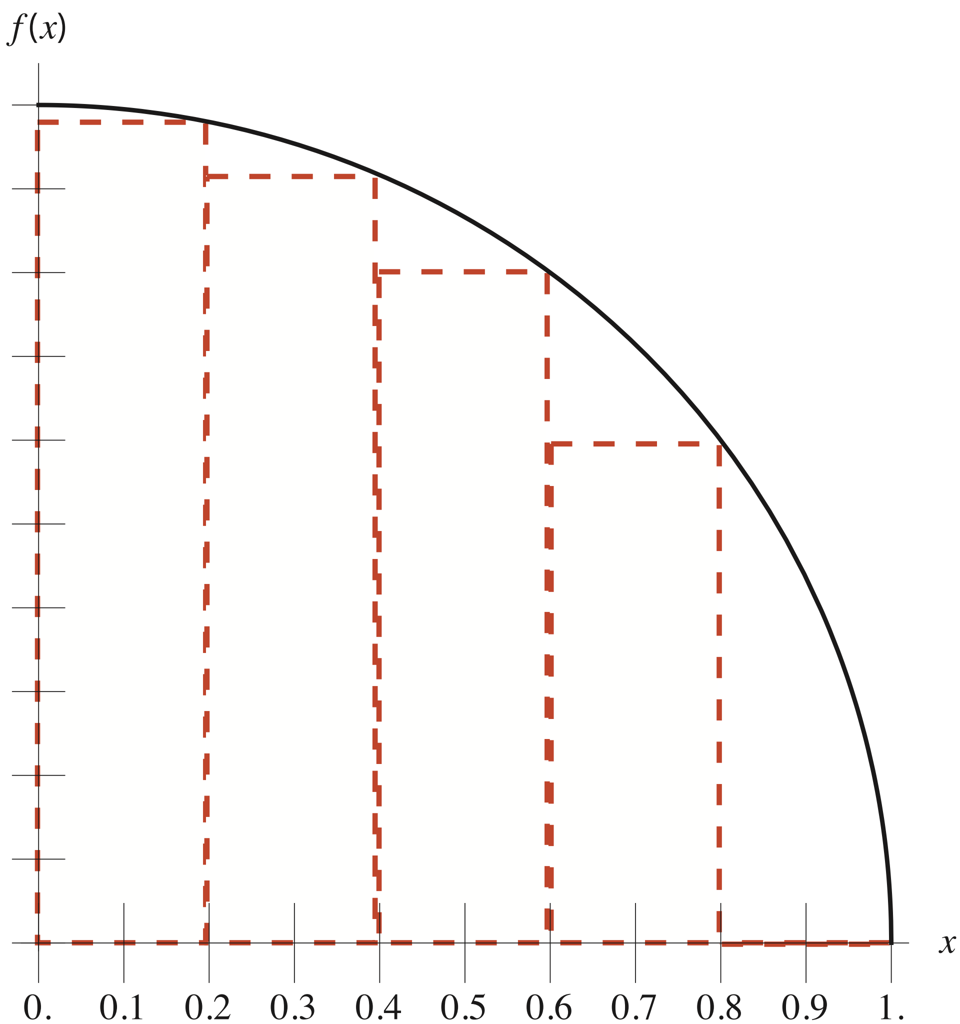
(b) Call your above approximation LEFT(5). Arrange the following in ascending order:

$$\text{LEFT}(5) \qquad\qquad A$$

$$A \; < \; \text{LEFT}(5)$$

because the above rectangles overshoot the curve, and therefore give an overestimate to the actual area.

2. Below again is a graph of the function $f(x) = \sqrt{1 - x^2}$.



(a) On top of this graph, draw in the rectangles that would represent a *right endpoint* Riemann sum approximation, with $n = 5$, to the area A. See above.

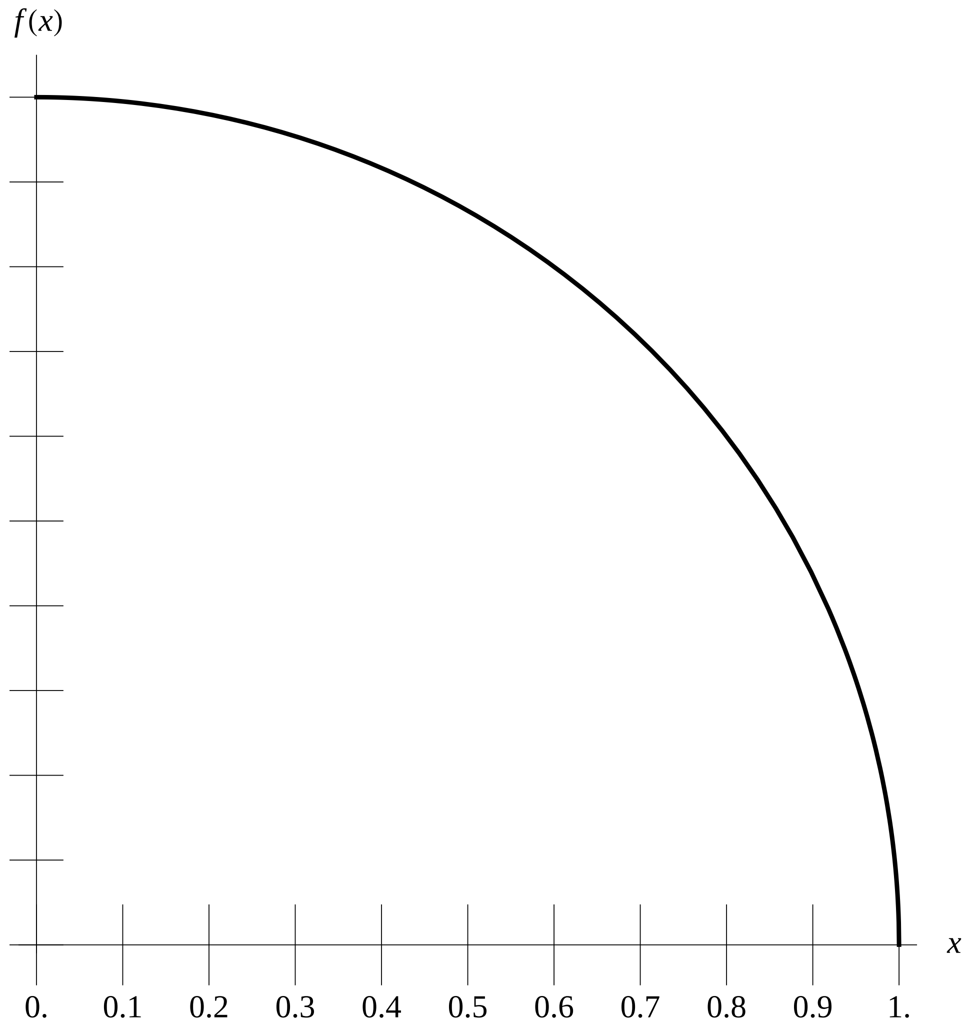
(b) Call your above approximation RIGHT(5). Arrange the following in ascending order:

RIGHT(5) LEFT(5) A

$$\text{RIGHT}(5) \; < \; A \; < \; \text{LEFT}(5)$$

because the above rectangles undershoot the curve, and therefore give an underestimate to the actual area.

3. We could also do a left, or a right, endpoint Riemann sum approximation with 100 rectangles. You don't have to draw these rectangles, but imagine what they might look like.



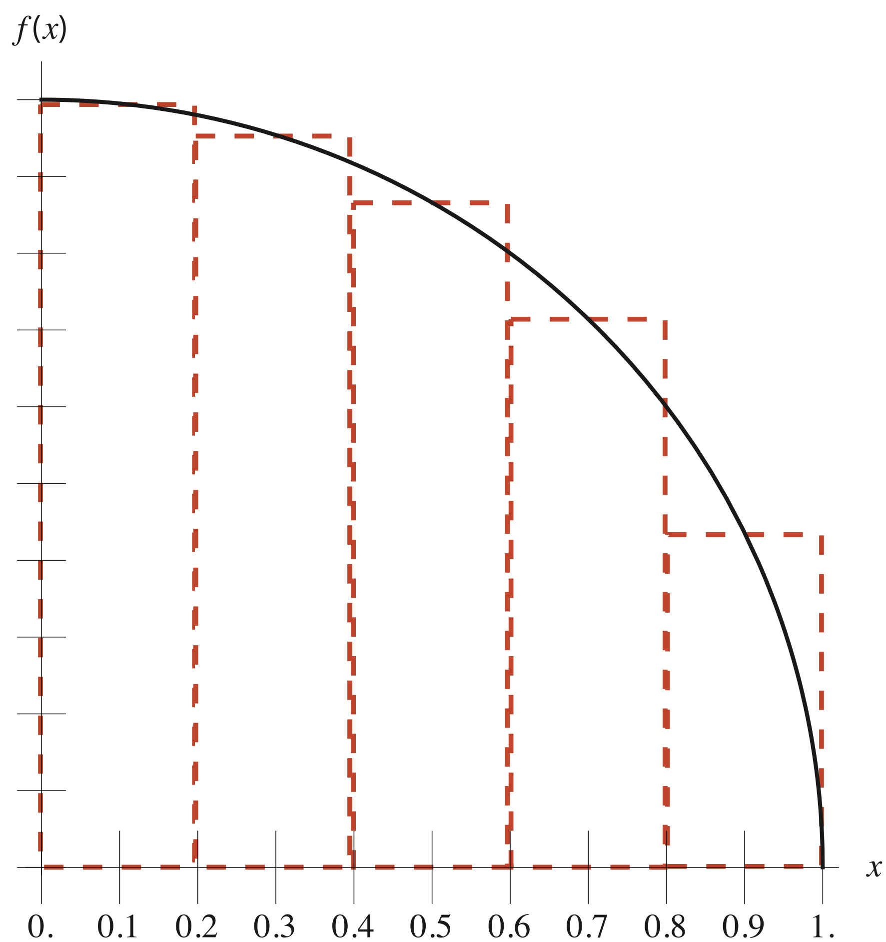
Call your left and right endpoint approximations with 100 rectangles LEFT(100) and RIGHT(100), respectively. Arrange the following in ascending order:

LEFT(100) RIGHT(100) RIGHT(5) LEFT(5) A

$$\text{RIGHT}(5) < \text{RIGHT}(100) < A < \text{LEFT}(100) < \text{LEFT}(5)$$

because, as with $n = 5$, we'd still expect right endpoint rectangles to underestimate and left endpoint rectangles to overestimate. However, using more rectangles gives better approximations, so the Riemann sums with 100 rectangles will be closer to the true area A than those with 5 rectangles.

4. Below again is a graph of the function $f(x) = \sqrt{1 - x^2}$.



(a) On top of this graph, draw in the rectangles that would represent a *midpoint* Riemann sum approximation, with $n = 5$, to the area A. See above.

(b) Call your above approximation MID(5). Arrange the following in ascending order:

MID(5) RIGHT(5) LEFT(5) A

$$\text{RIGHT}(5) < A < \text{MID}(5) < \text{LEFT}(5)$$

We see that the midpoint rectangles partially undershoot and partially overshoot the curve. So we'd expect to the midpoint rectangles to give better approximations than left or right endpoint rectangles. Also, if you look carefully at the rightmost rectangle above, you can convince yourself that the area of the overshoot there is a bit more than the area of the undershoot, so all in all, the midpoint rectangles overestimate by just a bit.

5. CHALLENGE: Arrange all of these in ascending order.

   MID(100)   MID(5)   LEFT(100)   RIGHT(100)   RIGHT(5)   LEFT(5) A

   By exercise 3 above, we have

   $$\text{RIGHT}(5) < \text{RIGHT}(100) < A < \text{LEFT}(100) < \text{LEFT}(5)$$

   The only thing missing here is MID(5). By exercise 4 above, MID(5) is somewhere between A and LEFT(5). So the only question is where MID(5) lands with respect to LEFT(100). Now 100 is a lot of rectangles, so let's guess that LEFT(100) gives us a better approximation that MID(5). Then we have

   $$\text{RIGHT}(5) < \text{RIGHT}(100) < A < \text{LEFT}(100) < \text{MID}(5) < \text{LEFT}(5)$$

6. What is A *exactly* (rounded to five decimal places)?

   A is the area of a quarter of a circle of radius 1, so

$$A = \frac{1}{4} \cdot \pi(1)^2 = \frac{\pi}{4} \approx 0.78540.$$

7. Check your answers, by actually computing all of the quantities in the above CHALLENGE question (using RIEMANN.sws, for example).

$$\text{RIGHT}(5) < \text{RIGHT}(100) < A < \text{LEFT}(100) < \text{MID}(5) < \text{LEFT}(5)$$

$$0.65926 < 0.78010 < 0.78540 < 0.79010 < 0.79300 < 0.85926$$