5.3 Separation of Variables

One of the principal uses of integration techniques is to find closed form solutions to differential equations. If you look back at the methods we have developed so far in this chapter, they are all applicable to differential equations of the form $y' = f(t)$ for some function f – that is, to differential equations where *the rate at which the* ***dependent*** *variable changes is a function of the* ***independent*** *variable only.* Such differential equations are often called "pure-time" equations, because they express rates of change purely in terms of the independent variable, which is frequently a time variable. See, for instance, Examples 5.1.4 and 5.2.2(b) above.

For pure-time initial value problems, we only need to find an antiderivative F for f, choose the constant C to satisfy the initial value, and we have our solution.

As we saw in earlier chapters, though, the behavior of y' often depends on the values of the *dependent* variable y, rather than the independent variable t – think of the SIR model, the exponential and logistic growth models, predator-prey problems, and so on. When y' is expressed purely in terms of y, we often call our differential equation "autonomous." In this section, we will see how our earlier techniques can be adapted to apply to autonomous equations, and to various differential equations of "mixed" type, where dy/dx is expressed in terms of both y and x.

The separation of variables procedure

The following example illustrates the general ideas.

Example 5.3.1. Solve the initial value problem

$$\frac{dy}{dx} = xy^2, \qquad y(1) = 3. \tag{5.3.1}$$

Solution. In the previous section, we employed the useful "trick" of treating a derivative du/dx like a fraction, so that we could manipulate the so-called "differentials" du and dx separately. Let's see how this idea can also help us in the present context.

We begin by rearranging the differential equation in (5.3.1) so that all y's (including the dy) are on the left-hand side, and all x's (including the dx) are on the right. To do this, we multiply both sides of (5.3.1) by dx, and divide both sides by y^2, to get

$$y^{-2}\,dy = x\,dx. \tag{5.3.2}$$

Placing an indefinite integral sign in front of the quantity on each side of (5.3.2) then gives

$$\int y^{-2}\,dy = \int x\,dx. \tag{5.3.3}$$

We perform the indicated antidifferentiations, to get

$$-y^{-1} + C_1 = \frac{x^2}{2} + C_2, \tag{5.3.4}$$

where C_1 and C_2 are (perhaps different) constants. We subtract C_1 from both sides of (5.3.4) to get

$$-y^{-1} = \frac{x^2}{2} + C, \tag{5.3.5}$$

where we have denoted the constant $C_2 - C_1$ simply by C.

The next step is to solve for C. We do so by substituting the initial condition $y(1) = 3$ (which says: when $x = 1$, $y = 3$) into (5.3.5), to get

$$\begin{aligned} -3^{-1} &= \frac{1^2}{2} + C \\ -\frac{1}{3} &= \frac{1}{2} + C \\ C &= -\frac{1}{3} - \frac{1}{2} = -\frac{2+3}{6} = -\frac{5}{6}. \end{aligned}$$

Putting this back into (5.3.5) gives

$$-y^{-1} = \frac{x^2}{2} - \frac{5}{6} = \frac{3x^2-5}{6}. \tag{5.3.6}$$

We obtained a common denominator on the right, because this will facilitate the final step, which is to solve for y. We do so by multiplying both sides of (5.3.6) by -1, and then taking reciprocals:

$$\begin{aligned} y^{-1} &= \frac{-3x^2+5}{6} \\ y &= \frac{6}{-3x^2+5}. \end{aligned} \tag{5.3.7}$$

In many previous examples involving antidifferentiation, we have checked our work by differentiating. We can do that here, too. However, the process is a bit more complicated now, because our derivative was given to us originally in terms of both x and y. One way to proceed, in such a case, is to express *both* sides of the given differential equation in terms of x alone, and to check that these two expressions in x really are equal.

Specifically: let y be given by (5.3.7). Then on the one hand, we have

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx}\left[\frac{6}{-3x^2+5}\right] \\ &= 6\frac{d}{dx}\left[(-3x^2+5)^{-1}\right] = 6\cdot(-(-3x^2+5)^{-2})\cdot\frac{d}{dx}\left[-3x^2+5\right] = 36x(-3x^2+5)^{-2}, \end{aligned} \tag{5.3.8}$$

while on the other hand, we have

$$xy^2 = x\left(\frac{6}{-3x^2+5}\right)^2 = \frac{36x}{(-3x^2+5)^2}. \tag{5.3.9}$$

The quantities in (5.3.8) and (5.3.9) are equal, so our function y of (5.3.6) indeed does satisfy the differential equation in (5.3.1).

To check that this function y also satisfies the initial condition in (5.3.1) is easier: we have

$$y(1) = \frac{6}{-3 \cdot 1^2 + 5} = \frac{6}{2} = 3,$$

so our initial condition is satisfied, and we are done with our check.

In the above example, we followed a "separate, integrate, evaluate, solve" strategy. This is a general approach, which we will summarize below, to solving initial value problems of the form

$$\frac{dy}{dx} = f(x)g(y), \qquad y(x_0) = y_0. \tag{5.3.10}$$

Note that the derivative in (5.3.10) takes a very special form: it's equal to a function in x *times* a function of y. If the derivative is not of that form (or can't be put into that form), then this approach won't work.

Here's an outline of this approach.

Step 1 (separate). In the differential equation, put all x's on the right and all y's on the right, to get $(g(y))^{-1}\,dy = f(x)\,dx$.

Step 2 (integrate). Add integral signs to both sides, to get $\int (g(y))^{-1}\,dy = \int f(x)\,dx$. Then perform the required integration, remembering the "$+C$" on the right.

Step 3 (evaluate). Substitute the initial condition $y(x_0) = y_0$ into the result from Step 2, to evaluate (that is, solve for) the constant C. Put this value of C back into the Step 2 result.

Step 4 (solve). Solve the result of Step 3 for y.

Separation of variables strategy for initial value problems of the form (5.3.10)

Of course, this strategy relies on finding manageable antiderivatives of $(g(y))^{-1}$ and $f(x)$, and as we've seen, this is not always possible. But it's possible often enough for the method to be useful.

Like integration by substitution, separation of variables entails manipulation of "differentials" like dx. (See Step 1 of the above boxed strategy.) At the end of this subsection, we'll justify this kind of manipulation in the present setting, as we did in the previous section for the substitution setting.

Example 5.3.2. The differential equation $dy/dt = kt$. We know from our studies in Section 3.1 that the exponential function $y = y_0 e^{kt}$ is *the* solution to the initial value problem

$$\frac{dy}{dt} = ky, \qquad y(0) = y_0. \tag{5.3.11}$$

Let's put aside this knowledge for a moment, and rediscover this solution using our new method.

To simplify our discussions, let's assume that we need to solve this initial value problem only for positive values of y. This is a reasonable assumption, since the equation $y' = ky$ typically models growth of some quantity for which negative values are not realistic. (In the exercises, we'll look at the more general case.)

We proceed according to the above boxed strategy:

$$\begin{aligned} \frac{dy}{y} &= k\,dt \\ \int \frac{dy}{y} &= \int k\,dt \\ \ln(|y|) &= kt + C \\ \ln(y) &= kt + C. \end{aligned} \tag{5.3.12}$$

The last step is because, again, we are assuming that $y > 0$; so $|y| = y$.

Into equation (5.3.12) we substitute the initial condition $y(0) = y_0$, to get

$$\ln(y_0) = k \cdot 0 + C = C,$$

so again by (5.3.12),

$$\ln(y) = kt + \ln(y_0).$$

To solve for y, we exponentiate both sides:

$$\begin{aligned} e^{\ln(y)} &= e^{kt+\ln(y_0)} = e^{kt}e^{\ln(y_0)} \\ y &= y_0 e^{kt}. \end{aligned} \tag{5.3.13}$$

We have thus solved the exponential growth initial value problem "from scratch." This is quite different from what we did in Section 3.1 – there, we *proposed* the solution $y = y_0 e^{kt}$, and then verified that it worked. Generally speaking, in "real life," one is not supplied with even a proposed solution; typically, one has to generate one, as we did in the above example.

That example illustrates the separation of variables technique in an *autonomous* context, meaning, again, where a derivative is expressed solely in terms of the *dependent* variable. If on the other hand, our differential equation is *pure-time* – the derivative is expressed solely in terms of the *independent* variable – then the separation of variables method is not required; see, for instance, Examples 5.1.4 and 5.2.2(b) above. It's worth noting, though, that separation of variables *may* be applied in these cases. We illustrate this in part (a) of the following example.

Example 5.3.3. Use separation of variables to solve:

(a) The initial value problem

$$\frac{dy}{dx} = 4x^3, \qquad y(-1) = 5.$$

(b) The differential equation

$$\frac{dy}{dx} = \frac{\cos(x)}{e^y};$$

(c) The initial value problem

$$\frac{ds}{dt} = e^{\sin(t)}\cos(t)\sqrt{s}, \qquad s(0) = 9.$$

Solution. (a) Applying the boxed procedure above, we have

$$\begin{aligned} dy &= 4x^3\,dx \\ \int dy &= \int 4x^3\,dx \\ y &= x^4 + C. \end{aligned}$$

Our initial condition then gives us $5 = y(-1) = (-1)^4 + C = 1 + C$, so $C = 5 - 1 = 4$, so

$$y = x^4 + 4.$$

(b) Our boxed procedure still applies when there is no initial condition provided, except that we ignore Step 3. Thus:

$$\begin{aligned} e^y\,dy &= \cos(x)\,dx \\ \int e^y\,dy &= \int \cos(x)\,dx \\ e^y &= \sin(x) + C \\ y &= \ln(\sin(x) + C). \end{aligned}$$

(c) This example demonstrates how separation of variables can sometimes be used in conjunction with integration by substitution:

$$\begin{array}{l|l} \dfrac{ds}{dt} = e^{\sin(t)}\cos(t)\sqrt{s} & \\ s^{-1/2}\,ds = e^{\sin(t)}\cos(t)\,dt & \\ \displaystyle\int s^{-1/2}\,ds = \int e^{\sin(t)}\cos(t)\,dt & u = \sin(t) \\ \displaystyle\int s^{-1/2}\,ds = \int e^u\,du & du = \cos(t)\,dt \\ 2s^{1/2} = e^u + C = e^{\sin(t)} + C. & \end{array}$$

Putting in $s(0) = 9$ gives

$$2 \cdot 9^{1/2} = e^{\sin(0)} + C = e^0 + C = 1 + C,$$

so $C = 2 \cdot 9^{1/2} - 1 = 6 - 1 = 5$, so

$$\begin{aligned} 2s^{1/2} &= e^{\sin(t)} + 5 \\ s^{1/2} &= \frac{e^{\sin(t)} + 5}{2} \\ s &= \left(\frac{e^{\sin(t)} + 5}{2}\right)^2. \end{aligned}$$

Some previously considered contexts

With the method of separation of variables, we can obtain formulas for solutions to a number of differential equations that were previously accessible only by Euler's method. One of the advantages of a *formula* is that it allows us to see how the parameters in the problem affect the solution.

We'll look at two problems. First, we'll show how the method can help us better understand the "supergrowth" phenomenon that we encountered in Exercise 5, Section 3.3. Then, using separation of variables, substitution, and a significant amount of algebra, we'll derive the formula (3.5.6) for logistic growth.

Example 5.3.4. Supergrowth. In Exercise 5, Section 3.3, we modeled the growth of a population Q by the initial value problem

$$\frac{dQ}{dt} = kQ^{1.2}, \qquad Q(0) = A.$$

To get a formula for the solution, we separate variables in the usual way:

$$\begin{aligned} \frac{dQ}{dt} &= k\,Q^{1.2} \\ \frac{dQ}{Q^{1.2}} &= k\,dt \\ \int Q^{-1.2} dQ &= \int k\,dt \\ \frac{Q^{-0.2}}{-0.2} &= kt + C. \end{aligned}$$

Applying the initial condition $Q(0) = A$ gives

$$\frac{A^{-0.2}}{-0.2} = k \cdot 0 + C = C,$$

so:

$$\begin{aligned} \frac{Q^{-0.2}}{-0.2} &= kt + \frac{A^{-0.2}}{-0.2} \\ Q^{-0.2} &= -0.2kt + A^{-0.2} \\ Q &= \left(-0.2kt + A^{-0.2}\right)^{1/(-0.2)} = \left(-0.2kt + A^{-1/5}\right)^{-5} \\ &= \left(-\frac{1}{5}kt + \frac{1}{\sqrt[5]{A}}\right)^{-5}. \end{aligned}$$

This formula for Q shows explicitly how the parameters k and A affect the solution. In particular: since a negative power of a quantity becomes infinite as that quantity approaches zero, our formula tell us that Q becomes infinite when $-kt/5 + 1/\sqrt[5]{A} = 0$, which is to say

$$t = \frac{5}{k\sqrt[5]{A}}.$$

The name *supergrowth* reflects the fact that a population growing according to this model will, in theory, “blow up” after a finite period of time. (Note that the length of this period is *inversely* proportional to the parameter k, and to the fifth root of the initial population.)

Example 5.3.5. Logistic growth. Here we derive “from scratch” the solution to the logistic growth initial value problem

$$R' = kR\left(1 - \frac{R}{b}\right), \qquad R(0) = R_0.$$

To simplify our derivations, we’ll assume that the population R is *less than* the carrying capacity b. (It may be shown, though, that the assumption $R > b$ yields exactly the same result.)

We begin in the usual way:

$$\begin{aligned} \frac{dR}{dt} &= kR\left(1 - \frac{R}{b}\right) \\ \frac{dR}{R\left(1 - \frac{R}{b}\right)} &= k\,dt \\ \int \frac{dR}{R\left(1 - \frac{R}{b}\right)} &= \int k\,dt. \end{aligned} \tag{5.3.14}$$

We now employ an algebra trick: we note that

$$R\left(1 - \frac{R}{b}\right) = R\left(R \cdot \left(\frac{1}{R} - \frac{1}{b}\right)\right) = R^2\left(\frac{1}{R} - \frac{1}{b}\right),$$

so that (5.3.14) yields

$$\int \frac{dR}{R^2\left(\frac{1}{R} - \frac{1}{b}\right)} = \int k\,dt.$$

We proceed to perform the integration on both sides; the integral on the left requires a substitution:

$$\int \frac{dR}{R^2\left(\frac{1}{R}-\frac{1}{b}\right)} = \int k\,dt \qquad\qquad u = \frac{1}{R}-\frac{1}{b}$$

$$\int \frac{-du}{u} = \int k\,dt \qquad\qquad du = -\frac{1}{R^2}\,dR$$

$$-\ln(|u|) = kt + C \qquad\qquad -du = \frac{1}{R^2}\,dR$$

$$-\ln\left(\left|\frac{1}{R}-\frac{1}{b}\right|\right) = kt + C.$$

Now we are assuming that R is less than the carrying capacity b; this implies (assuming $R > 0$, which is a reasonable assumption) that $1/R > 1/b$, so $1/R - 1/b > 0$. So we may remove the absolute value symbol above, to get

$$-\ln\left(\frac{1}{R}-\frac{1}{b}\right) = kt + C. \tag{5.3.15}$$

The initial condition $R(0) = R_0$ gives

$$-\ln\left(\frac{1}{R_0}-\frac{1}{b}\right) = k\cdot 0 + C = C;$$

putting this back into (5.3.15) then gives us

$$-\ln\left(\frac{1}{R}-\frac{1}{b}\right) = kt - \ln\left(\frac{1}{R_0}-\frac{1}{b}\right).$$

We now perform a rather lengthy series of manipulations to solve for R:

$$\begin{aligned}
-\ln\left(\frac{1}{R}-\frac{1}{b}\right) &= kt - \ln\left(\frac{1}{R_0}-\frac{1}{b}\right)\\
\ln\left(\frac{1}{R}-\frac{1}{b}\right) &= -kt + \ln\left(\frac{1}{R_0}-\frac{1}{b}\right)\\
\frac{1}{R}-\frac{1}{b} &= e^{-kt}\left(\frac{1}{R_0}-\frac{1}{b}\right)\\
\frac{1}{R} &= \frac{1}{b} + e^{-kt}\left(\frac{1}{R_0}-\frac{1}{b}\right) = \frac{R_0+(b-R_0)e^{-kt}}{R_0 b}\\
R &= \frac{R_0 b}{R_0+(b-R_0)e^{-kt}}.
\end{aligned}$$

(To get from the first line to the second, we multiplied through by -1. To get the third line, we exponentiated both sides of the second, using properties of exponentials. To get the fourth line, we added $1/b$ to both sides of the third, and then obtained a common denominator of $R_0 b$. The last line then simply entailed taking reciprocals.) This is precisely the result that was claimed in (3.5.6).

Justification

Each of our separation of variables arguments has begun in the following fashion:

$$\frac{dy}{dx} = f(x)g(y) \tag{a}$$
$$(g(y))^{-1}\,dy = f(x)\,dx \tag{b}$$
$$\int (g(y))^{-1}\,dy = \int f(x)\,dx. \tag{c}$$

In making such an argument, we've pretended, much as we did in integrating by substitution, that the so-called "differentials" dx and dy represent actual quantities. But again, they don't; they are just piece of the symbol dy/dx. So why do arguments like the above work?

The answer is that the middle statement (b) is just a heuristic, written to help guide us from (a) to (c). But in fact, we don't really need (b). This is because (c) follows from (a) in another way, that does not entail the differentials that appear in (b).

Here's how: assume that (a) is true. Let F be an antiderivative of f, so that the right-hand side of equation (c) equals $F(x) + C$. If we can show that the left-hand side of (c) equals the same thing, then the two sides of (c) will be equal, and we'll be done.

So we need only show that

$$\int (g(y))^{-1}\,dy = F(x) + C,$$

which is the same, by the boxed statement at the bottom of page 242, as showing that

$$\frac{d}{dx}\left[\int (g(y))^{-1}\,dy\right] = F'(x). \tag{5.3.16}$$

We demonstrate the latter by the chain rule:

$$\frac{d}{dx}\left[\int (g(y))^{-1}\,dy\right] = \frac{d}{dy}\left[\int (g(y))^{-1}\,dy\right]\frac{dy}{dx} = (g(y))^{-1}\frac{dy}{dx} = f(x) = F'(x)$$

(the second-to-last equality is by equation (a)), and we're done.

Exercises

Part 1: The separation of variables method

1. Use the method of separation of variables to find a formula for the solution of the differential equation $dy/dt = (y+5)^2 e^t$. Your formula should contain an arbitrary constant to reflect the fact that many functions solve the differential equation.

2. Use the method of separation of variables to find formulas for the solutions to the following initial value problems.

(a) $dy/dt = 1/y^2, \quad y(0) = 1.$

(b) $dz/dx = 3(x-2), \quad z(4) = 2.$

(c) $dy/dx = x^2/y^2, \quad y(3) = -1.$

(d) $dy/dx = \cos^2(y)e^x$, $y(0) = \pi/4$.
(Hints: $1/\cos^2(y) = \sec^2(y)$, and an antiderivative of $\sec^2(y)$ is $\tan(y)$.)

(e) $du/dv = v(1+u^2), \quad u(2) = 3.$

Part 2: Diffusion across a cell membrane

Consider a cell in some ambient environment (say, the bloodstream). Suppose that environment contains a substance (a medication, a chemical, etc.) that can pass through the cell membrane.

Let $K(t)$ denote the concentration of the substance *inside* the cell, as a function of time. (We might measure $K(t)$ in milligrams per liter, for example, and t in minutes.) Any change in $K(t)$ is called *diffusion*. We will model diffusion with the initial value problem

$$\frac{dK}{dt} = \beta(\gamma - K), \qquad K(0) = K_0, \qquad \textbf{(DM)}$$

for certain *positive* parameters β and γ, and K_0. The goal of the following exercises is to both *interpret* this initial value problem, and to use separation of variables to *solve* it.

3. Which of the constants β, γ, K_0 in **(DM)** denotes the initial concentration of the substance inside the cell?

4. Consider the following statement: "The concentration K of the substance inside the cell changes at a rate proportional to the *difference* between the concentration of the substance *outside* the cell, and the concentration inside the cell. (Also, we assume that the concentration outside the cell doesn't change.)" Given that this is correct, and is reflected by **(DM)**, answer these questions.

(a) Which of the constants β, γ, K_0 in **(DM)** denotes the concentration of the substance outside the cell?

(b) Which of the constants β, γ, K_0 in **(DM)** does it make most sense to call the *diffusion rate*?

5. When does diffusion happen more rapidly: when internal and external concentrations are very different, or when they're about the same? Please explain, in terms of **(DM)**.

6. Assume that the initial concentration of the substance inside the cell is **less** than the concentration outside the cell. Initially, is $K(t)$ increasing or decreasing? Please explain. (Refer to **(DM)** to answer, and remember that, by assumption, $\beta > 0$.)

7. We now use separation of variables to solve the above initial value problem **(DM)**.

(a) Separate variables in the differential equation in **(DM)**, and then take integrals on both sides, to show that

$$\int \frac{dK}{\gamma - K} = \beta t + C,$$

where C is some constant. We haven't yet evaluated the integral on the left, but we will now:

(b) In your answer to part (a) directly above, perform the integration in K, to deduce that

$$-\ln(|\gamma - K|) = \beta t + C.$$

Hint: into the integral in K, substitute $u = \gamma - K$. What is du for this substitution? (Remember that γ is a *constant* here.)

(c) Let's now assume, as we did in Exercise 6 above, that the concentration inside the cell starts out less than that outside. It's reasonable to also assume, then, that the internal concentration will *never* exceed the external concentration.

Explain why this assumption allows us to remove the absolute value symbols in your answer to part (b) of this exercise, so that

$$-\ln(\gamma - K) = \beta t + C.$$

(d) Plug the initial condition from **(DM)** into the result of part (c) above, solve for C, and put this value of C back into part (c), to show that

$$-\ln(\gamma - K) = \beta t - \ln(\gamma - K_0).$$

(e) Solve the result of part (d) of this exercise for K, to show that

$$K = \gamma - e^{-\beta t}(\gamma - K_0).$$

Some hints for the algebra: (i) multiply both sides of the result of part (d) of this exercise by -1; (ii) exponentiate both sides of your result; (iii) simplify what you get on the right, using the fact that

$$e^{-\beta t+\ln(\gamma-K_0)} = e^{-\beta t}e^{\ln(\gamma-K_0)} = (\gamma - K_0)e^{-\beta t};$$

and finally (iv) solve the result for K.

8. What do you get if you plug $t = 0$ into your formula for K from Exercise 7(e) above? Your answer should be a single parameter. What part of the initial value problem **(DM)** does this result agree with?

9. What does K approach as t approaches $+\infty$ in your solution to Exercise 7(e) above? Your answer should be a single parameter. (Hint: as $t \to +\infty$, $e^{-\beta t}$ approaches zero (since $\beta > 0$).) Why does your answer here make sense, in terms of the "real-world" situation at hand in this exercise?

Part 3: Various applications and contexts

10. **Newton's law of cooling.** According to Newton's law of cooling, in a room where the ambient temperature is A, the temperature Q of a hot object will change according to the differential equation

$$\frac{dQ}{dt} = -k(Q - A).$$

The constant k gives the rate at which the object cools.

(a) Find a formula for the solution to this equation using the method of separation of variables. Your formula should contain an arbitrary constant. **Note:** the process here, and the end result, are quite similar to those of Exercises 3–10 above.

(b) Suppose A is 20°C and k is .1° per minute per °C. If time t is measured in minutes, and $Q(0) = 90$°C, what will Q be after 20 minutes?

(c) How long does it take for the temperature to drop to 30°C?

11. (a) Suppose a cold drink at 36°F is sitting in the open air on a summer day when the temperature is 90°F. If the drink warms up at a rate of .2°F per minute per °F of temperature difference, write a differential equation to model what will happen to the temperature of the drink over time.

(b) Obtain a formula for the temperature of the drink as a function of the number of minutes t that have passed since its temperature was 36°F.

(c) What will the temperature of the drink be after 5 minutes; after 10 minutes?

(d) How long will it take for the drink to reach 55°F?

12. **A leaking tank**. One may show that the differential equation

$$\frac{dV}{dt} = -k\sqrt{V}$$

models the volume $V(t)$ of water in a leaking tank, after t hours.

(a) Use the method of separation of variables to show that

$$V(t) = \frac{k^2}{4}(C - t)^2$$

is a solution to the differential equation, for any value of the constant C.

(b) Explain why the function

$$V(t) = \begin{cases} \frac{k^2}{4}(C - t)^2 & \text{if } 0 \le t \le C, \\ 0 & \text{if } C < t. \end{cases}$$

is *also* a solution to the differential equation. Why is *this* solution more relevant to the leaking tank problem than the solution in part (a)?

13. **A falling body with air resistance**. The differential equation

$$\frac{dv}{dt} = -g - bv$$

may be to model the motion of a body falling under the influence of gravity (g, a constant) and air resistance (bv). Here b is a positive constant, and v is the velocity of the body at time t.

(a) Solve the differential equation by separating variables, and obtain

$$v(t) = \frac{1}{b}\left(Ce^{-bt} - g\right),$$

where C is an arbitrary constant.

(b) Now impose the initial condition $v(0) = 0$ (so the body starts its fall from rest) to determine the value of C. What is the formula for $v(t)$ now?

(c) The distance $x(T)$ that the body has fallen by time T is given by the integral

$$x(T) = \int_0^T v(t)\,dt, \quad \text{because} \quad \frac{dx}{dt} = v \quad \text{and} \quad x(0) = 0.$$

Use your formula for $v(t)$ from part (b) to find $x(T)$.

14. (a) **Supergrowth**. In Example 5.3.4 above, we analyzed the initial value problem

$$\frac{dQ}{dt} = kQ^p, \qquad Q(0) = A$$

when $p = 1.2$. (Of course, we've also studied the case $p = 1$, which corresponds to exponential growth.) Find a formula for the solution $Q(t)$ when $p = 2$. Your answer should be expressed in terms of the growth constant k and the initial population size A.

(b) Your formula in part (a) should demonstrate that Q becomes infinite at some finite time $t = \tau$. When is τ? Again, your answer should involve both A and k.

15. **General supergrowth**. Find the solution to the initial value problem

$$\frac{dQ}{dt} = kQ^p, \qquad Q(0) = A$$

for *any* value of the exponent p, other than $p = 1$ (which gives exponential growth). For which values of p does Q blow up to ∞ at a finite time $t = \tau$? What is τ (in terms of k, p, and A)?

16. **Logistic growth revisited.** In Example 5.3.5 above, we saw that the solution to the logistic growth initial value problem

$$R' = kR\left(1 - \frac{R}{b}\right), \qquad R(0) = R_0$$

is given by the formula

$$R(t) = \frac{R_0 b}{R_0 + (b - R_0)e^{-kt}}. \tag{$*$}$$

(a) Put $t = 0$ into $(*)$ to find $R(0)$. (Recall that $e^0 = 1$.) Explain how your result agrees with what you already knew (from the initial value problem itself).

(b) Use $(*)$ to determine what happens to $R(t)$ as time evolves indefinitely – that is, as $t \to +\infty$. (Hint: as $t \to +\infty$, e^{-kt} approaches zero.) Your answer should involve a single parameter.

Explain how your result agrees with what you already knew about carrying capacity.