Chapter 5

Techniques of Integration

Chapter 4 introduced the integral. There it was defined numerically, as the limit of approximating Riemann sums. Evaluating integrals by applying this basic definition tends to take a long time if a high level of accuracy is desired. If one is going to evaluate integrals at all frequently, it is thus important to find **techniques of integration** for doing this efficiently. For instance, if we evaluate a function at the midpoints of the subintervals, we get much faster convergence than if we use either the right or left endpoints of the subintervals.

A powerful class of techniques is based on the observation made at the end of Chapter 4, where we saw that The Fundamental Theorem of Calculus gives us a second way to find an integral, using antiderivatives. While a Riemann sum will usually give us only an approximation to the value of an integral, an antiderivative will give us the exact value. The drawback is that antiderivatives often can't be expressed in **closed form** – that is, as a **formula** in terms of named functions. Even when antiderivatives can be so expressed, the formulas are often difficult to find. Nevertheless, such a formula can be so powerful, both computationally and analytically, that it is often worth the effort needed to find it. In this chapter, we will explore several techniques for finding the antiderivative of a function given by a formula.

5.1 Antiderivatives

Definition

Recall that we say F is an **antiderivative** of f if $F' = f$. Here are some examples.

FUNCTION:	x^2	$1/y$	$\sin(u)$	$2\cos(t)\sin(t)$	2^z
	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$
ANTIDERIVATIVE:	$\dfrac{x^3}{3}$	$\ln y$	$-\cos(u)$	$\sin^2(t)$	$\dfrac{2^z}{\ln 2}$

Notice that you go up (↑) from the bottom row to the top by carrying out a differentiation. To go down (↓) you must "undo" that differentiation. The process of reversing, or undoing, a differentiation is called **antidifferentiation**. You should differentiate each function on the bottom row to check that it is an antiderivative of the function above it.

While a function can have only one derivative, it has many antiderivatives. For example, the functions $1 - \cos(u)$ and $99 - \cos(u)$ are also antiderivatives of the function $\sin(u)$, since

$$\frac{d}{du}[1 - \cos(u)] = \sin(u) = \frac{d}{du}[99 - \cos(u)].$$

In fact, every function $F(u) = C - \cos(u)$ is an antiderivative of $f(u) = \sin(u)$, for any constant C whatsoever. This observation is true in general. That is, if F is an antiderivative of a function f, then so is $F + C$, for any constant C. This follows from the addition rule for derivatives, because if $F' = f$, then

$$(F + C)' = F' + C' = F' + 0 = f;$$

that is, $(F + C)' = f$ as well.

Remark 5.1.1. It is tempting to claim the converse – that *every* antiderivative of f is equal to $F+C$, for some appropriately chosen value of C. In fact, you will often see this statement written. The statement is true, though, only for continuous functions – functions with no breaks in their domains. If the function f does have breaks, then there will be more antiderivatives than those of the form $F + C$ for a *single* constant C. Instead, over each piece of the domain of f, F can be modified by a *different* constant and still yield an antiderivative for f. Exercises 12 and 13 at the end of this section explore this for a couple of cases. If f is continuous, though, $F + C$ will cover all the possibilities, and we sometimes say that $F + C$ is *the* antiderivative of f. For the sake of keeping a compact notation, we will even write this when the domain of f consists of more than one interval. You should understand, though, that in such cases, over each piece F can be modified by a different constant.

For future reference, we collect a list of basic functions whose antiderivatives we already know. Remember that each antiderivative in the table can have an arbitrary constant added to it.

function $f(x)$	antiderivative $F(x)$
x^p	$\frac{x^{p+1}}{p+1}, \quad p \neq -1$
$\sin(ax)$	$-\frac{\cos(ax)}{a}, \quad a \neq 0$
$\cos(ax)$	$\frac{\sin(ax)}{a}, \quad a \neq 0$
e^{ax}	$\frac{e^{ax}}{a}, \quad a \neq 0$
b^x	$\frac{b^x}{\ln b}, \quad b > 0$
$\frac{1}{x}$	$\ln(\lvert x \rvert)$

All of these antiderivatives are easily verified, by differentiating the function F on the right-hand side of any given row, and checking that you get the corresponding function on the left. For example,

$$\frac{d}{dx}\left[-\frac{\cos(ax)}{a}\right] = -\frac{1}{a}\big(-a\sin(ax)\big) = \sin(ax),$$

which verifies the second row of the table.

The last row of the table merits some explanation. It wouldn't be correct to say that $\ln(x)$ is always an antiderivative of $1/x$: for one thing, $\ln(x)$ is not even defined for $x < 0$, while $1/x$ is. We'd like to find a function F such that $F'(x) = 1/x$ whenever $1/x$ makes sense. We claim that $F(x) = \ln(|x|)$ does the job. To see this, note first that $F(x)$ does make sense for all $x \neq 0$, just as $1/x$ does. Next, we differentiate $F(x)$ by considering two cases:

(i) If $x > 0$, then $|x|$ is the same as x, so

$$\frac{d}{dx}[\ln(|x|)] = \frac{d}{dx}[\ln(x)] = \frac{1}{x},$$

as desired. Next,

(ii) If $x < 0$, then $|x|$ is the same as $-x$, so

$$\frac{d}{dx}[\ln(|x|)] = \frac{d}{dx}[\ln(-x)] = \frac{1}{-x}\frac{d}{dx}[-x] = \frac{1}{-x}\cdot(-1) = \frac{1}{x}.$$

Together, (i) and (ii) tell us that $d[\ln(|x|)]/dx = 1/x$ whenever $x \neq 0$, and this confirms the last row of our table.

There are a couple of other functions that don't appear in the above table, but whose antiderivatives are often needed:

function	antiderivative
$\frac{1}{\sqrt{1-x^2}}$	$\arcsin(x)$
$\frac{1}{1+x^2}$	$\arctan(x)$

The antiderivatives here are inverse trigonometric functions. The implied differentiation formulas

$$\frac{d}{dx}[\arcsin(x)] = \frac{1}{\sqrt{1-x^2}} \quad \text{and} \quad \frac{d}{dx}[\arctan(x)] = \frac{1}{1+x^2}$$

were derived in Section 3.6. (See Example 3.6.2, and Exercise 11 of that section.)

Notation

As we've noted previously – see Example 4.5.2(v) – there are functions that, even though they are expressible in terms of familiar quantities, do not have antiderivatives that can be written in closed form. This is not necessarily to say that these functions don't *have* antiderivatives.

To see this, let $p(t)$ be any function that is not too "weird" on an interval $[a,b]$. (A function that is continuous on this interval – that is, essentially, it has no breaks or jumps there – is good enough.) Consider the accumulation function

$$E(T) = \int_a^T p(t)\, dt.$$

As we've seen in Section 4.2, we have $E'(T) = p(T)$ for any number T between a and b. That is: E *is* an antiderivative of p for such values of T.

The point is that the definite integral gives us a means of *defining* antiderivatives. For example,

$$F(T) = \int_0^T e^{-t^2/2}\, dt \tag{5.1.1}$$

is an antiderivative of $f(t) = e^{-t^2/2}$, even though, again, there's no "closed" formula for this antiderivative. (The formula (5.1.1) is not considered "closed" because it requires the integral sign.)

The connection between antiderivatives and integrals is so pervasive that the integral sign – with the "limits of integration" omitted – is also used to denote an antiderivative:

> **Notation**: The most general antiderivative of f is denoted $\displaystyle\int f(x)\, dx$.

Remark 5.1.2. By "the most general antiderivative of f," we mean "the *set* of all possible antiderivatives of f." So strictly speaking, $\int f(x)\, dx$ denotes not a single function, but a set of functions. Generally speaking, though, we can find all elements of this set by just finding one element, and then adding an "arbitrary constant" $+C$ to that single element. See Remark 5.1.1 above.

With this new notation, the antiderivatives we have listed so far can be written in the following form.

$$\int x^p\,dx = \frac{x^{p+1}}{p+1} + C \qquad (p \neq -1)$$
$$\int \sin(ax)\,dx = -\frac{\cos(ax)}{a} + C \qquad (a \neq 0)$$
$$\int \cos(ax)\,dx = \frac{\sin(ax)}{a} + C \qquad (a \neq 0)$$
$$\int e^{ax}\,dx = \frac{e^{ax}}{a} + C \qquad (a \neq 0)$$
$$\int b^x\,dx = \frac{b^x}{\ln b} + C \qquad (b > 0)$$
$$\int \frac{1}{x}\,dx = \ln(|x|) + C$$
$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin(x) + C$$
$$\int \frac{1}{1+x^2}\,dx = \arctan(x) + C$$

The integral sign $\int$ now has two distinct meanings. Originally, it was used to describe the *number*

$$\int_a^b f(x)\,dx,$$

which is a signed area, or a limit of a sequence of Riemann sums. Because this integral has a definite numerical value, it is called the **definite integral**. In its new meaning, the integration sign is used to describe the antiderivative

$$\int f(x)\,dx,$$

which is a *function* (really, a *set* of functions), not a number. To contrast the new use of $\int$ with the old, and to remind us that the new expression is a variable quantity, it is called the **indefinite integral**. The function that appears in either a definite or an indefinite integral is called the **integrand**. The terms "antiderivative" and "indefinite integral" are completely synonymous. We will tend to use the former term in general discussions, using the latter term when focusing on the process of finding the antiderivative.

Because an indefinite integral represents an antiderivative, the process of finding an antiderivative is sometimes called **integration**. We've also used this term to designate the process of finding a definite integral. Thus the term *integration*, as well as the symbol for it, has two distinct meanings.

Using Antiderivatives

According to the fundamental theorem, we can use an *indefinite* integral to find the value of a *definite* integral – and this largely explains the importance of antiderivatives. In the language of indefinite integrals, the statement of the fundamental theorem in the box on page 229 takes the following form.

$$\int_a^b f(x)\,dx = F(b) - F(a), \text{ where } F(x) = \int f(x)\,dx.$$

Example 5.1.1. Find $\displaystyle\int_1^4 x^2\,dx$.

Solution. We have

$$\int x^2\,dx = \frac{1}{3}x^3 + C.$$

It follows that

$$\int_1^4 x^2\,dx = \left(\frac{1}{3}x^3 + C\right)\Big|_1^4 = \left(\frac{1}{3}\times 4^3 + C\right) - \left(\frac{1}{3}\times 1^3 + C\right) = \frac{64}{3} + C - \frac{1}{3} - C = 21.$$

Note that, in the above example, the two appearances of "$+C$" cancel each other. This cancellation will occur no matter what function we are integrating, since

$$(F(x)+C)\Big|_a^b = (F(b)+C) - (F(a)+C) = F(b) + C - F(a) - C = F(b) - F(a) = F(x)\Big|_a^b.$$

This implies that it does not matter which value of C we choose to do the calculation. Usually, we just take $C = 0$ (which amounts to the procedure we followed in Section 4.5).

Example 5.1.2. Find $\displaystyle\int_0^{\pi/2} \cos(t)\,dt$.

Solution. This time, the indefinite integral we need is

$$\int \cos(t)\,dt = \sin(t) + C.$$

The value of the definite integral is therefore

$$\int_0^{\pi/2} \cos(t)\,dt = \sin(t)\Big|_0^{\pi/2} = \sin\pi/2 - \sin 0 = 1 + C - 0 - C = 1.$$

Finding Antiderivatives

What we have seen above is this:

The statement $F'(x) = f(x)$ is *the same* as the statement $\int f(x)\,dx = F(x) + C$.

Relationship between indefinite integrals and derivatives

Because of this fact, we can verify many statements about antidifferentiation by considering the corresponding differentiation facts.

In particular, the following basic indefinite integral rules can be verified using analogous rules for derivatives.

$$\int kf(x)\,dx = k\int f(x)\,dx \qquad \textbf{(constant multiple rule);}$$
$$\int (f(x)+g(x))\,dx = \int f(x)\,dx + \int g(x)\,dx \qquad \textbf{(sum rule).}$$

(Here, k is a constant.)

For example, the sum rule for indefinite integrals may be demonstrated as follows. Let F be an antiderivative for f, and G an antiderivative for g. Then by the above boxed statement,

$$\int f(x)\,dx + \int g(x)\,dx = (F(x)+C_1) + (G(x)+C_2) = F(x) + G(x) + C, \tag{5.1.2}$$

where C_1 and C_2 are arbitrary constants, and $C = C_1 + C_2$. (Since C_1 and C_2 can be anything, so can C; so C is an arbitrary constant as well.) But by the sum rule for differentiation, the derivative of the right-hand side of (5.1.2) equals $F'(x) + G'(x) = f(x) + g(x)$, so again by the above boxed statement,

$$\int (f(x)+g(x))\,dx = F(x) + G(x) + C. \tag{5.1.3}$$

The right-hand sides of equations (5.1.2) and (5.1.3) are equal, so the left-hand sides are equal too, and this is what we wanted to show.

Example 5.1.3. This example illustrates the use of both the addition and the constant multiple rules.

$$\begin{aligned}\int (7e^x + \cos(x))\,dx &= \int 7e^x\,dx + \int \cos(x)\,dx \\ &= 7\int e^x\,dx + \int \cos(x)\,dx \\ &= 7e^x + \sin(x) + C.\end{aligned}$$

The next example illustrates how initial value problems of the form

$$F'(x) = f(x), \qquad F(0) = y_0 \tag{5.1.4}$$

may be solved using indefinite integrals.

Example 5.1.4. Find a function F such that

$$F'(x) = 3x^2 - \sin(x), \qquad F(0) = 7.$$

Solution. We are looking for a function F of x whose derivative is a given function of x, and whose value at a certain point is a given number. The general strategy for such problems is to first find *any and all* functions with the indicated derivative, and then to select, among all of those functions, the one that satisfies the given initial condition.

In other words, the first step is to find the *most general antiderivative*, which is to say the *indefinite integral*, of the given function. That is: the equation $F'(x) = 3x^2 - \sin(x)$ tells us that

$$F(x) = \int (3x^2 - \sin(x))\, dx = x^3 + \cos(x) + C. \tag{5.1.5}$$

Now, we need only figure out what C is.

To do so, we substitute the condition $F(0) = 7$ into equation (5.1.5), to get

$$7 = F(0) = 0^3 + \cos(0) + C = 1 + C.$$

Solving for C gives $C = 7 - 1 = 6$. We plug this value of C back into (5.1.5) to get our complete solution:

$$F(x) = x^3 + \cos(x) + 6.$$

Note that the above example, and the kind of problem described by (5.1.4), represent a very special type of initial value problem, where a derivative is expressed exclusively in terms of the *independent* variable. We've previously encountered examples where a derivative is expressed exclusively in terms of the *dependent* variable (and some parameters) – for example, the exponential growth equation $P' = kP$, and the the logistic equation $P' = kP(1 - P/b)$. In Section 5.3, we'll examine certain situations where both independent and dependent variables are involved in the formula for the derivative.

Also in the following sections, we will develop the antidifferentiation rules that correspond to the chain rule and to the product rule. They are called *integration by substitution* and *integration by parts*, respectively.

Because indefinite integrals are often difficult to calculate, reference manuals in mathematics and science often include tables of integrals. There are sometimes many hundreds of individual formulas, organized by the type of function being integrated.

Computers are having a major impact on integration techniques. And computer software packages that can find any existing formula for a definite integral have become widespread, and have had a profound impact on the importance of integration techniques. Just as hand-held calculators have rendered obsolete many traditional arts, like using logarithms for performing multiplications or knowing how to interpolate in trig tables, so have computers hastened a decrease in emphasis on humans' fluency with integration techniques. While some will continue to derive pleasure from becoming proficient in these skills, for most users it will generally be much faster, and more accurate, to use an appropriate software package. Nevertheless, for those going on in mathematics and the sciences, it will still to be useful to be able to perform some of the simpler integrations by hand reasonably rapidly. And perhaps more importantly, some experience with concrete integral computations helps create a solid foundation for our understanding of the ideas and abstractions.

The subsequent sections of this chapter develop the most commonly needed techniques employed for such computations.

Exercises

Part 1: Basic antidifferentiation

1. Find a formula for each of the following indefinite integrals. For each integral, verify that your result is correct by differentiation. **Example:**

$$\int 4\cos(3x+2)\,dx = \frac{4}{3}\sin(3x+2)+C.$$

Verification:

$$\frac{d}{dx}\left[\frac{4}{3}\sin(3x+2)+C\right] = \frac{4}{3}\frac{d}{dx}\big[\sin(3x+2)\big]+0 = \frac{4}{3}\cdot\cos(3x+2)\cdot 3 = 4\cos(3x+2). \quad \checkmark$$

(a) $\int 3x\,dx$

(b) $\int 3u\,du$

(c) $\int e^z\,dz$

(d) $\int (5t^4 + 5\cdot 4^t)\,dt$

(e) $\int \left(7y + \frac{1}{y}\right)dy$

(f) $\int \left(7y - \frac{4}{y^2}\right)dy$

(g) $\int (5\cos(w) - \cos(5w))\,dw$

(h) $\int dx$ (This just means $\int 1\,dx$.)

(i) $\int e^{z+2}\,dz$

(j) $\int \cos(4x-2)\,dx$

(k) $\int \frac{5}{1+r^2}\,dr$

(l) $\int \frac{1}{1+4s^2}\,ds$ (Hint: guess and check, using the fact that $d[\arctan(s)]/ds = 1/1+s^2$.)

(m) $\int (2x+3)^7\,dx$

(n) $\int \cos(1-x)\,dx$

2. Verify, by differentiation, that the antiderivatives given in the list on page 241 are correct.

3. Find $\int (a+by)\,dy$, where a and b are constants.

Part 2: Initial value problems

4. (a) Solve the initial value problem

$$F'(x) = 7, \qquad F(0) = 12.$$

(b) Solve the initial value problem

$$G'(x) = 7, \qquad G(3) = 1.$$

(c) Do $F(x)$ and $G(x)$ differ by a constant? If so, what is the value of that constant?

5. (a) Find an antiderivative $F(t)$ of $f(t) = t + \cos(t)$ for which $F(0) = 3$.

(b) Find an antiderivative $G(t)$ of $f(t) = t + \cos(t)$ for which $G(\pi/2) = -5$.

(c) Do $F(t)$ and $G(t)$ differ by a constant? If so, what is the value of that constant?

6. Solve the initial value problem

$$\frac{dy}{dx} = \frac{1}{1+x^2}, \qquad y(0) = 4.$$

7. Solve the initial value problem

$$\frac{dp}{dq} = 4q - \frac{3}{q^2}, \qquad p(1) = 6.$$

Part 3: Guessing an checking (and checking and guessing)

For part (a) of each of the exercises below, note that the instruction "Verify that $F(x)$ is an antiderivative of $f(x)$" simply means "show that $F'(x) = f(x)$." Subsequent parts of these exercises ask you to either to (i) elaborate on your answer from part (a), or (ii) to adjust your answer from part (a), to obtain a function that has the indicated derivative.

8. (a) Verify that $(1 + x^3)^{10}$ is an antiderivative of $30x^2(1 + x^3)^9$.

(b) Find an antiderivative of $x^2(1 + x^3)^9$.

(c) Find an antiderivative of $x^2 + x^2(1 + x^3)^9$.

9. (a) Verify that $x \ln(x)$ is an antiderivative of $1 + \ln(x)$.

(b) Find an antiderivative of $\ln(x)$. (Do you see how you can use part (a) to find this antiderivative?)

10. Recall that $F(y) = \ln(y)$ is an antiderivative of $1/y$ for $y > 0$. According to the text, *every* antiderivative of $1/y$ over this domain must be of the form $\ln(y) + C$ for an appropriate value of C.

(a) Verify that $G(y) = \ln(2y)$ is also an antiderivative of $1/y$.

(b) Find C so that $\ln(2y) = \ln(y) + C$.

11. (a) Verify that $-\cos^2(t)$ is an antiderivative of $2\cos(t)\sin(t)$.

(b) Since you've already seen that $\sin^2(t)$ is an antiderivative of $2\cos(t)\sin(t)$ (see the discussion at the very beginning of this section), you should be able to show that

$$-\cos^2(t) = \sin^2(t) + C$$

for an appropriate value of C. What is C?

Part 4: Miscellaneous

12. The function $\ln(|x|) + C$ is an antiderivative of $1/x$, for any constant C, but there are more antiderivatives. This can happen because the domain of $1/x$ is broken into two parts. To see this, let

$$G(x) = \begin{cases} \ln(-x) & \text{if } x < 0, \\ \ln(x) + 1 & \text{if } x > 0. \end{cases}$$

(a) Explain why there is no value of C for which

$$\ln(|x|) + C = G(x).$$

This shows that the functions $\ln(|x|) + C$ do not exhaust the set of antiderivatives of $1/x$.

(b) Construct two more antiderivatives of $1/x$ and sketch their graphs. What is the general form of the new antiderivatives you have constructed? (A suggestion: you should be able to use two separate constants C_1 and C_2 to describe the general form.)

13. In the list on page 241, the antiderivative of x^p is given as

$$\frac{1}{p+1}x^{p+1} + C.$$

For some values of p this is correct, with only a single constant C needed. For other values of p, though, the domain of x^p will consist of more than one piece, and $\frac{1}{p+1}x^{p+1}$ can be modified by a different constant over each piece. For what values of p does this happen?

14. Find $F'(x)$ for the following functions. In parts (a), (b), and (d) do the problems two ways: by finding an antiderivative, and by using The Fundamental Theorem to get the answer without evaluating an antiderivative. Check that the answers agree.

(a) $F(x) = \displaystyle\int_0^x (t^2 + t^3)\, dt.$

(b) $F(x) = \int_1^x \frac{1}{u}\, du.$

(c) $F(x) = \int_1^x \frac{v}{1+v^3}\, dv.$

(d) $F(x) = \int_0^{x^2} \cos(t)\, dt.$

(e) $F(x) = \int_1^{x^2} \frac{v}{1+v^3}\, dv.$ [Hint: let $u = x^2$ and use the chain rule.]

Comment: It may seem that parts (c) and (e) are more difficult than the others. However, there is a way to apply the fundamental theorem of calculus here to get answers to parts (c) and (e) quickly and with little effort.

15. Consider the two functions

$$F(x) = \sqrt{1+x^2} - 1 \quad \text{and} \quad G(x) = \int_0^x \frac{t}{\sqrt{1+t^2}}\, dt.$$

(a) Show that F and G both satisfy the initial value problem

$$y' = \frac{x}{\sqrt{1+x^2}}, \qquad y(0) = 0.$$

(b) Since an initial value problem typically has a *unique* solution, F and G should be equal. Assuming this, determine the exact value of the following definite integrals.

$$\int_0^1 \frac{t}{\sqrt{1+t^2}}\, dt, \qquad \int_0^2 \frac{t}{\sqrt{1+t^2}}\, dt, \qquad \int_0^5 \frac{t}{\sqrt{1+t^2}}\, dt.$$

16. Find the area under the curve $y = x^3 + x$ for x between 1 and 4.

17. Find the area under the curve $y = e^{3x}$ for x between 0 and $\ln 3$.