4.3 Riemann Sums

In Example 4.1.5, we estimated energy consumption in a town by replacing the power function $p(t)$ by a step function. Let's pause to describe that process in somewhat more general terms that we can adapt to other contexts. The power graph, an approximating step function, and an energy estimate are shown below.

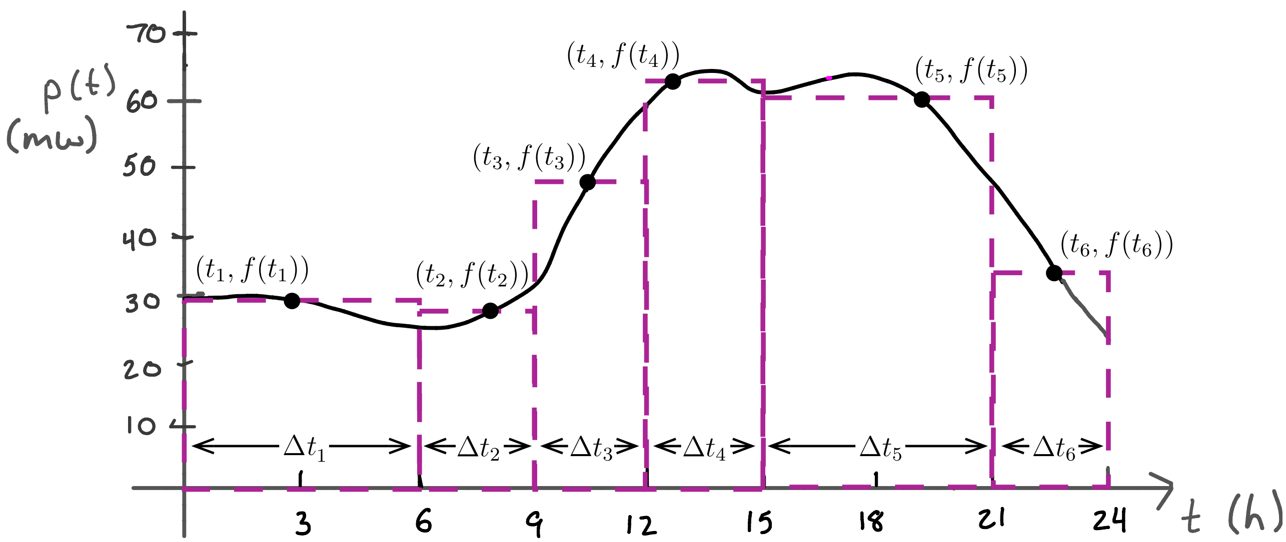

Figure 4.8. Power demand in a town over a 24-hour period, and an approximating step function

Using the above rectangles as approximations to the energy used on the corresponding subintervals, we obtain yet another estimate to the energy used over the 24 hours:

$$\begin{aligned} \text{energy used} &\approx p(t_1)\Delta t_1 + p(t_2)\Delta t_2 + \cdots + p(t_6)\Delta t_6 \\ &= 30 \times 6 + 29 \times 3 + 48 \times 3 + 63 \times 3 + 60 \times 6 + 33 \times 3 \\ &= 1{,}059 \text{ mwh.} \end{aligned} \tag{4.3.1}$$

The height of the first step is 30 megawatts. This is the *actual* power level at the time t_1 indicated on the graph. That is, $p(t_1) = 30$ megawatts. We found a power level of 30 megawatts by **sampling** the power function at the time t_1. The height of the first step could have been different if we had sampled the power function at a different time. In general, if we sample the power function $p(t)$ at the time t_1 in the interval Δt_1, then we would estimate the energy used during that time to be

$$\text{energy} \approx p(t_1)\Delta t_1 \text{ mwh.}$$

Notice that t_1 is not necessarily in the middle, or at either end, of the first interval. It is simply a time when the power demand is representative of what's happening over the entire interval.

We can describe what happens in the other time intervals the same way. If we sample the kth interval at the point t_k, then the height of the kth power step will be $p(t_k)$, and our *estimate* for the energy used during that time will be equal to the area of the kth rectangle:

$$\text{energy} \approx p(t_k)\Delta t_k \text{ mwh.}$$

We now have a general way to construct an approximation for the power function and an estimate for the energy consumed over a 24-hour period. It involves these steps.

1. Choose any number n of subintervals, and let them have arbitrary positive widths Δt_1, Δt_2, ..., Δt_n, subject only to the condition

$$\Delta t_1 + \cdots + \Delta t_n = 24 \text{ hours.}$$

2. Sample the kth subinterval at any point t_k, and let $p(t_k)$ represent the power level over this subinterval.

3. Estimate the energy used over the 24 hours by the sum

$$\text{energy} \approx p(t_1)\Delta t_1 + p(t_2)\Delta t_2 + \cdots + p(t_n)\Delta t_n \text{ mwh.}$$

The expression on the right-hand side is called a **Riemann sum** for the power function $p(t)$ on the interval $0 \le t \le 24$ hours. We've seen similar sums in the context of other accumulation function calculations.

The work of Bernhard Riemann (1826–1866) has had a profound influence on contemporary mathematics and physics. His revolutionary ideas about the geometry of space, for example, are the basis for Einstein's theory of general relativity.

The formal definition of Riemann sum, for an arbitrary function $f(x)$, is as follows.

Definition. Suppose the function $f(x)$ is defined for x in the interval $[a,b]$. Then a **Riemann sum** for $f(x)$ on $[a,b]$ is an expression of the form

$$f(x_1)\Delta x_1 + f(x_2)\Delta x_2 + \cdots + f(x_n)\Delta x_n.$$

The interval $[a,b]$ has been divided into n subintervals whose lengths are Δx_1, ..., Δx_n, respectively, and for each k from 1 to n, x_k is some point in the kth subinterval.

In other words: a Riemann sum for $f(x)$ is exactly the kind of sum we've been using to approximate accumulation functions for $f(x)$.

Notice that once the function and the interval have been specified, a Riemann sum is determined by the following data:

- A **decomposition** of the original interval into subintervals (which determines the lengths of the subintervals).
- A **sampling point** chosen from each subinterval (which determines a value of the function on each subinterval).

A Riemann sum for $f(x)$ is a sum of products of values of Δx and values of $y = f(x)$. If x and y have units, then so does the Riemann sum; its units are the units for x times the units for y. When a Riemann sum arises in a particular context, the notation may look different from what appears in the definition just given: the variable might not be x, and the function might not be $f(x)$. For example, the energy approximation we considered at the beginning of the section is a Riemann sum for the power demand function $p(t)$ on $[0, 24]$.

It is important to note that, from a mathematical point of view, a Riemann sum is just a number. It's the *context* that provides the meaning: Riemann sums for a power demand that varies over time approximate total energy consumption; Riemann sums for a speed that varies over time approximate total distance; Riemann sums for a force that varies over distance approximates total work done. And so on.

The enormous range of choices in this process means there are innumerable ways to construct a Riemann sum for a function $f(x)$. However, we are not really interested in *arbitrary* Riemann sums. On the contrary, we want to build Riemann sums that will give us good estimates for our accumulation functions. Therefore, we will choose each subinterval length Δx_k so small that the values of $f(x)$ over that subinterval differ only very little from the sampled value $f(x_k)$. A Riemann sum constructed with *these* choices will then differ only very little from the actual accumulation function under consideration.

Let's now consider a purely geometric application of Riemann sums.

Example 4.3.1. Approximate the area under the graph of $f(x) = \sqrt{x-1}$, using a *left endpoint Riemann sum approximation* with ten evenly spaced subintervals. By "left endpoint Riemann sum approximation" we mean: use the left endpoint of each subinterval as your sampling point in that interval.

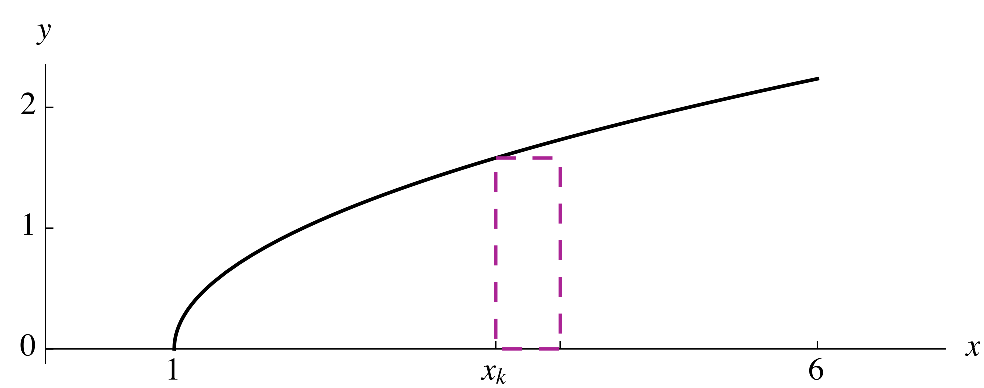

Figure 4.9. $f(x) = \sqrt{x-1}$ **and a representative rectangle**

Solution. To keep the picture relatively uncluttered, we've drawn in a single representative rectangle, in Figure 4.9 above, rather than all ten of the rectangles that will be used in the approximation.

We're told that all subinterval lengths Δx_1,Δx_2, ...,Δx_{10} should be the same. Let's denote this common length by Δx: we compute that

$$\Delta x = \frac{\text{length of the interval [1,6]}}{\text{number of subintervals}} = \frac{6-1}{10} = 0.5.$$

Also, we're doing a left endpoint approximation. So our first sampling point x_1 is the left endpoint of our first subinterval; that is, $x_1 = 1$. Our second sampling point x_2 is the left endpoint of the second subinterval; so $x_2 = 0.5$, and so on all the way up to our tenth subinterval, whose left endpoint is $x_{10} = 5.5$.

So our area, call it A, is approximated by the following Riemann sum:

$$\begin{aligned} A &\approx f(x_1)\Delta x_1 + f(x_2)\Delta x_2 + \cdots + f(x_{10})\Delta x_{10} \\ &= \Delta x(f(x_1) + f(x_2) + \cdots + f(x_{10})) \\ &= 0.5(f(1) + f(1.5) + \cdots + f(5.5)) \\ &= 0.5(\sqrt{1-1} + \sqrt{1.5-1} + \cdots + \sqrt{5.5-1}) \\ &= 0.5(0 + 0.707107 + 1 + 1.22474 + 1.41421 + 1.58114 + 1.73205 + 1.87083 + 2 + 2.12132) \\ &= 6.82570. \end{aligned} \tag{4.3.2}$$

Later, we'll see that the true area A equals 7.4536 (to four decimal places). So our approximation is an underestimate. This can can also be seen from Figure 4.9 above: as the representative rectangle illustrates, all of our left endpoint rectangles will *undershoot* the graph of $f(x)$. So the sum of their areas will be less than the area A we are trying to estimate.

More generally, as the above figure suggests, the following is true. On any interval $[a,b]$ where $f(x)$ is *increasing* (and non-negative), any left endpoint Riemann sum approximation to $f(x)$ will underestimate the area under the graph of $f(x)$, over that interval. Similar things happen when $f(x)$ is *decreasing* on an interval, and/or we use a *right endpoint* Riemann sum approximation. You should think about whether the Riemann sums give underestimates or overestimates in each of these cases.

The Sage program RIEMANN.sws, below, automates the left endpoint Riemann sum approximation of Example 4.3.1 above. The program has been written so that it may be easily modified, to accommodate different functions $f(x)$, different intervals $[a,b]$, different numbers n of rectangles, and even different varieties of sampling points (other than just left endpoints).

Program: RIEMANN.sws
Left endpoint Riemann sums, with evenly spaced intervals

```
var ('x')            # our variable is called x
f(x) = sqrt(x-1)     # this is where you put your function
a = 1
b = 6
n = 10
Deltax = (b-a)/n     # baselength of the rectangles
                     # The following formula gives you a left endpoint sum
RS = sum(f(a+(k-1)*Deltax)*Deltax for k in [1..n])
print round(RS,5)    #prints the output rounded to 5 decimal places
```

It's important to review the above program, to understand the purposes of the various lines. Such an understanding will make it easy to adapt the program to other situations.

The purpose of the first five lines is clear. The sixth line expresses the following fact: if the interval $[a,b]$ is to be divided into n subintervals of equal length Δx, then

$$\boxed{\Delta x = \frac{b-a}{n}}$$

Formula for baselength of rectangles, in a Riemann sum approximation where all rectangles have equal baselength

What does the line

`RS = sum(f(a+(k-1)*Deltax)*Deltax for k in [1..n])` (4.3.3)

signify? Well, the crucial thing to note here is that the quantity $a + (k-1)\Delta x$ in (4.3.3) *is* the left endpoint of the kth subinterval. To see this, think of it this way: to get to the left endpoint of the kth subinterval, you start at $x = a$, and perform $k - 1$ "jumps" of length Δx each.

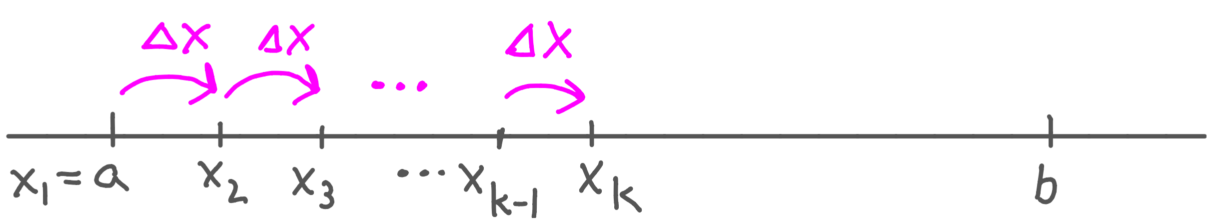

Figure 4.10. The "jumps" required to get to the left endpoint of the kth subinterval

So the line (4.3.3) is just saying "To get your approximation RS, add up the numbers $f(x_k)\Delta x$, where x_k is the left endpoint of the kth subinterval, for $1 \leq k \leq n$." And this is exactly the sum we want for our left endpoint Riemann sum approximation.

We summarize:

$$\textbf{Left endpoint Riemann sums: } x_k = a + (k-1)\Delta x$$

Formula for the sampling points x_k, in a left endpoint Riemann sum approximation (with all intervals of equal length)

By modifying RIEMANN.sws, you can calculate Riemann sums for other sampling points, other intervals, other numbers of rectangles, and other functions. For example, to sample at midpoints, you must start at $x = a$, and make $(k-1) + 1/2 = k - 1/2$ jumps each of length Δx. To sample at right endpoints, you again start at $x = a$, but this time make k such jumps. So we have the following formulas:

$$\textbf{Midpoint Riemann sums: } x_k = a + (k - 1/2)\Delta x$$
$$\textbf{Right endpoint Riemann sums: } x_k = a + k\Delta x$$

Formulas for the sampling points x_k, in midpoint and right endpoint Riemann sum approximations (with all intervals of equal length)

Of course, in a Riemann sum approximation, not all subintervals need to have the same length. We have already done a number of approximations where various different lengths were used in the same approximation. But equal lengths make for nicer formulas, and simpler code, and more systematic algorithms. We will use subintervals of equal length except where otherwise noted.

Exercises

Part 1: Riemann sums "by hand"

For the exercises in this part, follow the procedure of Example 4.3.1 above. You should **not** use the program RIEMANN.sws here.

You do not need to provide graphs of the functions in question, thought you may if you want. (A sketch, by hand or computer, with a representative rectangle (see Figure 4.9), might be helpful for you *own* reference, to help visualize what's going on.)

Of course, you can use a calculator for computations like those in the last two lines of (4.3.2).

1. (a) Approximate the area A under the graph of $f(x) = \sqrt{1 + x^3}$ on the interval [3,7], using a left endpoint Riemann sum approximation with four rectangles, all of equal baselength.

(b) Do you think your estimate is an overestimate or underestimate of the actual area A? Hint: you might want to sketch the function on this interval. Then consider the comments, following Example 4.3.1, about increase/decrease versus underestimates/overestimates.

(c) Repeat parts (a) and (b) of this exercise, but this time using a *right* endpoint Riemann sum approximation. (Everything else – the function, the interval, the number of rectangles – should remain the same as above.)

2. Repeat Example 4.3.1, but this time using a *right* endpoint Riemann sum approximation.

3. Repeat Example 4.3.1, but this time using a *mipoint* Riemann sum approximation.

Part 2: Using RIEMANN.sws

For the exercises in this part, *do not* try to write down, or compute, the requested Riemann sums by hand. Simply use the above program RIEMANN.sws, suitably modified.

4. Calculate left endpoint Riemann sums for the function $\sqrt{1+x^3}$ on the interval [3,7] using 10, 100, 1,000, and 10,000 equally-spaced subintervals. (You may need to be a bit patient when n is as large as 10,000.) Note that these Riemann sums approximations seem to be approaching a limit – that is, zeroing in on some particular number – as the number of subintervals gets larger and larger. What does that limit seem to be, rounded to the nearest hundredth?

5. Repeat Exercise 4 above for the same function, but this time, on the interval [1,3].

6. Using the results of Exercises 4 and 5 above, provide estimates of the area A under the graph of $f(x) = \sqrt{1+x^3}$, over each of the following intervals: (a) [1,3], (b) [3,7], and (c) [1,7]. (For part (c), think about how you can use the answers to parts (a) and (b).)

7. Repeat Exercise 4 above, but this time, use *right endpoint* Riemann sums.

8. (a) Repeat Exercise 4 above, but this time, use *midpoint* Riemann sums.

(b) Comment on the relative "efficiency" of midpoint Riemann sums, versus left and right endpoint Riemann sums (at least for the function $\sqrt{1+x^3}$ on the interval [3,7]). (A more efficient procedure is one that will "zero in" on a particular value faster than a less efficient one.)

9. Calculate left endpoint Riemann sums for the function

$$f(x) = \cos(x^2) \quad \text{on the interval } [0,4],$$

using 100, 1000, and 10000 equally-spaced subintervals.

10. Calculate right endpoint Riemann sums for the function

$$f(x) = \frac{\cos(x)}{1+x^2} \text{ on the interval } [2,3],$$

using 10, 100, and 1000 equally-spaced subintervals. The Riemann sums are all negative; why? (A suggestion: sketch the graph of f, preferably by computer. What does that tell you about the signs of the terms in a Riemann sum for f?)

11. (a) Calculate midpoint Riemann sums for the function

$$H(z) = z^3 \quad \text{on the interval } [-2, 2],$$

using 10, 100, and 1000 equally-spaced subintervals. The Riemann sums are all zero; why?

(b) Repeat part (a) using *left endpoint* Riemann sums. Are the results still zero? Can you explain the difference, if any, between these two results?

12. Using RIEMANN.sws, obtain a sequence of estimates for the area under each of the following curves. Continue until the first three decimal places stabilize in your estimates.

(a) $y = x^2$ over $[0, 1]$ (b) $y = x^2$ over $[0, 3]$ (c) $y = x\sin(x)$ over $[0, \pi]$

13. What is the area under the curve $y = \exp(-x^2)$ over the interval $[0, 1]$? Give an estimate that is accurate to three decimal places. Sketch the curve and shade the area.

Part 3: Making approximations

14. The aim of this question is to determine how much electrical energy was consumed in a house over a 24-hour period, when the power demand p was measured at different times to have these values:

time (24-hour clock)	power (watts)
1:30	275
5:00	240
8:00	730
9:30	300
11:00	150
15:00	225
18:30	1880
20:00	950
22:30	700
23:00	350

Notice that the time interval is from $t = 0$ hours to $t = 24$ hours, but the power demand was not sampled at either of those times.

(a) Set up an estimate for the energy consumption in the form of a Riemann sum $p(t_1)\Delta t_1 + \cdots + p(t_n)\Delta t_n$ for the power function $p(t)$. To do this, you must identify explicitly the value of n, the sampling times t_k, and the time intervals Δt_k that you used in constructing your estimate. [Note: the sampling times come from the table, but there is wide latitude in how you choose the subintervals Δt_k.]

(b) What is the estimated energy consumption, using your choice of data? There is no single "correct" answer to this question. Your estimate depends on the choices you made in setting up the Riemann sum.

(c) Plot the data given in the table in part (a) on a (t, p)-coordinate plane. Then draw on the same coordinate plane the step function that represents your estimate of the power function $p(t)$. The width of the kth step should be the time interval Δt_k that you specified in part (a); is it?

(d) Estimate the *average* power demand in the house during the 24-hour period.

Waste production. A colony of living yeast cells in a vat of fermenting grape juice produces waste products – mainly alcohol and carbon dioxide – as it consumes the sugar in the grape juice. It is reasonable to expect that another yeast colony, twice as large as this one, would produce twice as much waste over the same time period. Moreover, since waste accumulates over time, if we double the time period we would expect our colony to produce twice as much waste.

These observations suggest that waste production is proportional to both the size of the colony and the amount of time that passes. If P is the size of the colony, in grams, and Δt is a short time interval, then we can express waste production W as a function of P and Δt:

$$W = k \cdot P \cdot \Delta t \text{ grams.}$$

If Δt is measured in hours, then the multiplier k has to be measured in units of grams of waste per hour per gram of yeast.

The preceding formula is useful only over a time interval Δt in which the population size P does not vary significantly. If the time interval is large, and the population size can be expressed as a function $P(t)$ of the time t, then we can estimate waste production by breaking up the whole time interval into a succession of smaller intervals Δt_1, Δt_2, ..., Δt_n and forming a Riemann sum

$$k\, P(t_1)\, \Delta t_1 + \cdots + k\, P(t_n)\, \Delta t_n \approx W \text{ grams.}$$

The time t_k must lie within the time interval Δt_k, and $P(t_k)$ must be a good approximation to the population size $P(t)$ throughout that time interval.

15. Suppose the colony starts with 300 grams of yeast (i.e., at time $t = 0$ hours) and it grows exponentially according to the formula

$$P(t) = 300\, e^{0.2\, t}\,.$$

If the waste production constant k is 0.1 grams per hour per gram of yeast, estimate how much waste is produced in the first four hours. Use a Riemann sum with four hour-long time intervals and measure the population size of the yeast in the middle of each interval – that is, "on the half-hour."