Chapter 4

The Integral

There are many contexts – energy, work, area, volume, distance travelled, mass, and profit/loss are just a few – where the quantity in which we are interested is a product of known quantities. For example, the electrical energy needed to burn three 100 watt light bulbs for T hours is $300T$ watt-hours. The calculation becomes more complicated, though, if lights are turned off and on during the time interval of length T. We face the same complication in any context in which one of the factors in a product varies. To describe such a product we will introduce the **integral**.

As we will see, the integral itself can be viewed as a variable quantity. By analyzing the rate at which that quantity changes, we will find that every integral can be expressed as the solution to a particular differential equation. We will thus be able to use all our tools for solving differential equations to determine integrals.

4.1 Power and Energy

A power company charges customers for the work done by the electricity it supplies. The work done by electricity is usually referred to as **(electrical) energy**.

The rate at which electrical energy is supplied, or consumed, is referred to as *power*. In this section, we explore relationships between power and energy. The broad purpose of this exploration is to introduce the idea of an *accumulation* function, which generalizes the notion of a product to situations where one of the factors varies.

Ultimately, our goal is to understand how accumulation functions are related to derivatives, and to *areas*.

Part A: Power supplied at a constant rate

Suppose electrical energy is supplied, or consumed, at a **constant** rate over a given period of time. Again, this rate is called *power*. So in this situation, we have the formula

$$\text{energy} = \text{power} \times \text{elapsed time}, \tag{4.1.1}$$

where "elapsed time" refers to duration of the interval in question.

Example 4.1.1. A 60 watt bulb burning for two and a half hours consumes

$$60 \text{ watts} \times 2.5 \text{ hours} = 150 \text{ watt-hours}$$

of energy.

Note that we have used **watts**, also denoted w, as our units for power, and hours (h) as our units for time, whence our units for energy become watt-hours, or wh.

Keep in mind that power is a **rate**. It might be helpful to think of a watt as a "watt-hour per hour;" the word "per" helps remind us of rates of change.

Other units for electric power are the kilowatt (kw) (= 1,000 watts), the megawatt (mw) (= 1,000,000 watts), and the gigawatt (gw) (= 1,000,000,000 watts). So electrical *energy* is also measured in kilowatt-hours (kwh), in megawatt-hours (mwh), and in gigawatt-hours (gwh).

Energy is also sometimes measured in joules. By definition, 1 joule equals one thirty-six-hundredth of a watt-hour. Or, since there are 3600 seconds in an hour,

$$1 \text{ joule} = \frac{1}{3600} \text{ watt-hours} \times 3600 \ \frac{\text{seconds}}{\text{hour}} = 1 \text{ watt-second.}$$

To fully appreciate the relationship between energy and power, it will be instructive to revisit the light bulb of Example 4.1.1 above. But this time, we will consider cumulative energy consumption not only at the end of the 2.5 hour time period, but at any time T "along the way."

Example 4.1.2. What is the cumulative amount of energy consumed, call it $E(T)$, by the bulb of Example 4.1.1, from the beginning of the 2.5-hour period to time T, where T is any number of hours between 0 and 2.5?

Solution. By (4.1.1), the amount of energy consumed over the given interval of time is

$$E(T) = 60 \text{ watts} \times T \text{ hours} = 60T \text{ watt-hours.}$$

We've used a "T" rather than a "t" in this example to emphasize the fact that we are measuring a cumulative effect over time. (Capital letters seem somehow more cumulative than lower-case ones.)

To conclude this subsection, let's look more closely at the relationship between the above energy function $E(T)$ and the power demand $p(T)$ at time T, where T is, again, any number between 0 and 2.5. Of course, power demand is constant over this interval; we have $p(T) = 60$ (watts) for any such T. Here are the graphs of $p(T)$ and $E(T)$ over this period:

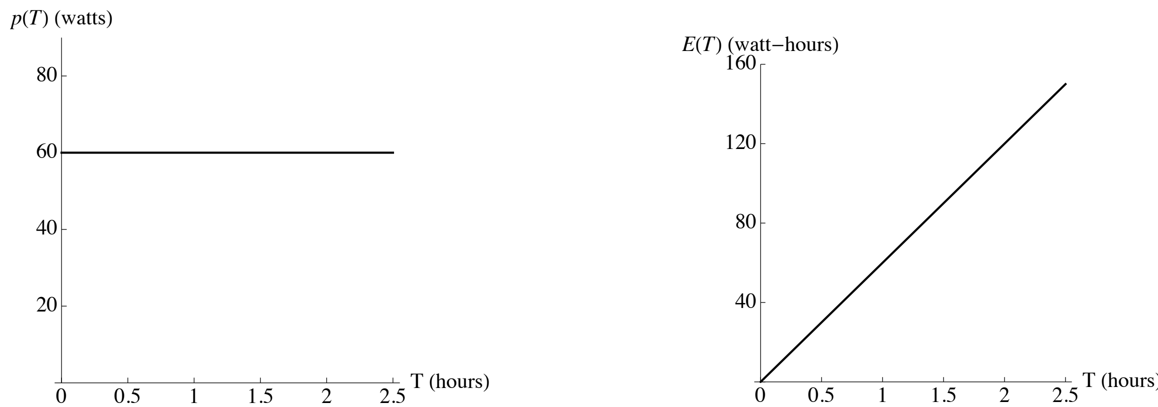


Figure 4.1. Power and energy, for an interval over which power demand is constant

The important thing to note from the above figure is that **the slope of the energy function** $E(T)$ **equals the height of the power function** $p(T)$ – both of these numbers equal 60 watts. But the height of the power function is just $p(T)$ (which is constant since power is being supplied at a constant rate), and the slope of the energy function is just $E'(T)$ (since the slope of a linear function equals the derivative of that function). So what we have just seen is that

$$E'(T) = p(T). \tag{4.1.2}$$

Of course, these arguments apply, so far, only to a situation where power is supplied at a constant rate. But similar conclusions may be drawn in more general situations. We will see this over the remainder of this section (and chapter).

Part B: Power that varies in steps

Consider the following modification of Example 4.1.1.

Example 4.1.3. A 3-way light bulb burns at 60 watts for 3.5 hours, at 100 watts for another 1.5 hours, and at 45 watts for another three hours. How much cumulative energy is consumed by the light bulb over the entire 8-hour period?

The key here is to use equation (4.1.1) in *stages*, and then to add up the results from those stages. That is: over the first 3.5 hours, 60 watts $\times$ 3.5 hours = 210 watt-hours of energy are consumed; over the next 1.5 hours, 100 watts $\times$ 1.5 hours = 150 watt-hours of energy are consumed; over the next 3 hours, 45 watts $\times$ 3 hours = 135 watt-hours of energy are consumed. So the total energy consumption, over the 8-hour period, is given by

$$\text{energy} = 210 + 150 + 135 = 495 \text{ watt-hours.}$$

Now, as we did in Part A of this section, let's ask a more general question.

Example 4.1.4. What is the cumulative energy consumption $E(T)$ of the above 3-way light bulb, in watts, T hours from the beginning of our six-hour period?

Solution. The answer is more complicated in the present case, because of the changing power demand. To address this, we consider three separate intervals: (i) $0 \le T \le 3.5$; (ii) $3.5 \le T \le 5$; and (iii) $5 \le T \le 8$. Note that power is *constant* on each of these intervals.

We argue as follows.

(i) For T in the first interval, power demand equals 60 watts, so by (4.1.1),

$$E(T) = 60T \text{ watt-hours} \quad \text{for } 0 \le T \le 3.5. \tag{4.1.3}$$

(ii) Next, suppose T is between 3.5 and 5 hours. Over the first 3.5 of these hours, $60 \times 3.5 = 210$ watt-hours are consumed, by (4.1.1). The remaining amount of time, between 3.5 hours and T hours, constitutes an interval of length $T - 3.5$ hours. By (4.1.1), then, $100 \times (T - 3.5) = 100T - 350$ watt-hours are consumed over this remaining period. So the *total* amount of energy consumed over the interval from 0 to T, in this case, is given by:

$$E(T) = 210 + (100T - 350) = 100T - 140 \text{ watt-hours} \quad \text{for } 3.5 \le T \le 5. \tag{4.1.4}$$

(iii) Finally, suppose T is between 5 and 8 hours. Over the first 5 of these hours, $100 \times 5 - 140 = 360$ watt-hours are consumed, by (4.1.4). The remaining amount of time, between 5 hours and T hours, constitutes an interval of length $T - 5$ hours. By (4.1.1), then, $45 \times (T - 5) = 45T - 225$ watt-hours are consumed over this remaining period. So the *total* amount of energy consumed over the interval from 0 to T, in this case, is given by:

$$E(T) = 360 + (45T - 225) = 45T + 135 \text{ watt-hours} \quad \text{for } 5 \le T \le 8. \tag{4.1.5}$$

Let's summarize: for the 3-way bulb of Example 4.1.3, the energy $E(T)$ consumed over the first T hours of the period in question, in watt-hours, is given by

$$E(T) = \begin{cases} 60T & \text{if } 0 \le T \le 3.5, \\ 100T - 140 & \text{if } 3.5 \le T \le 5, \\ 45T + 135 & \text{if } 5 \le T \le 8. \end{cases} \tag{4.1.6}$$

Here are graphs of the power function $p(T)$ and the energy function $E(T)$, for the above 3-way light bulb:

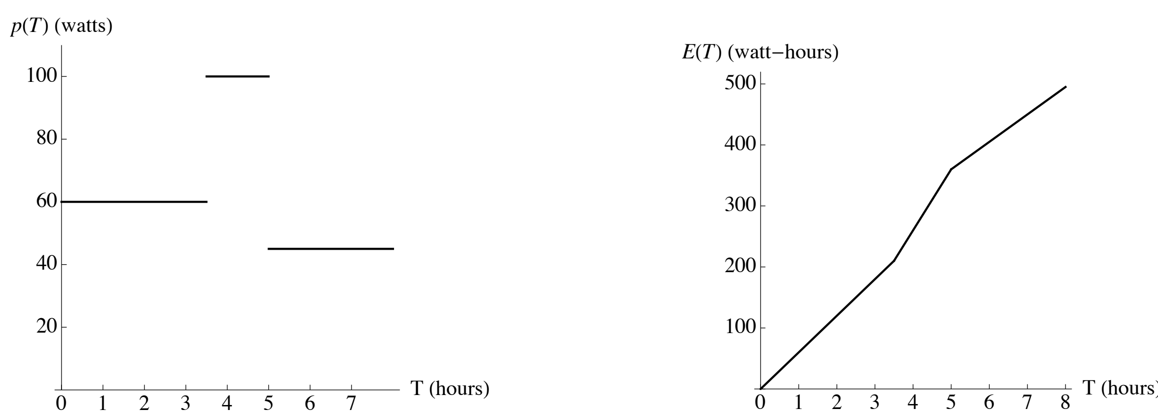


Figure 4.2. Power and energy, when power demand is "constant in steps"

As was the case for our 60-watt bulb of Part A of this section, we see here that, for our 3-way bulb, **the slope of the energy function** $E(T)$ **equals the height of the power function** $p(T)$. Actually, this is not *quite* true in the present case. The energy function $E(T)$ in Figure 4.2 above does not in fact *have* a slope at the T-values $T = 3.5$ and $T = 5$. (At these points, $E(T)$ has "corners," and is therefore not locally linear.)

But at any other point in between 0 and 8, we see that the slope of the $E(T)$ graph *does* equal the height of the $p(T)$ graph. (Between 0 and 3.5, the slope of $E(T)$ and the height of $p(T)$ are both equal to 60; between 3.5 and 5, this slope and this height are both equal to 100; between 5 and 8, these are both equal to 45.) In other words, except at the points where $E(T)$ is not locally linear, we have, again,

$$E'(T) = p(T). \tag{4.1.7}$$

We next consider an even more general situation.

Part C: Power that varies continuously

Suppose that power demand $p(t)$ of a certain town, over a 24 hour period – from midnight, denoted $t = 0$, to midnight, denoted $t = 24$ – is described by the following graph:

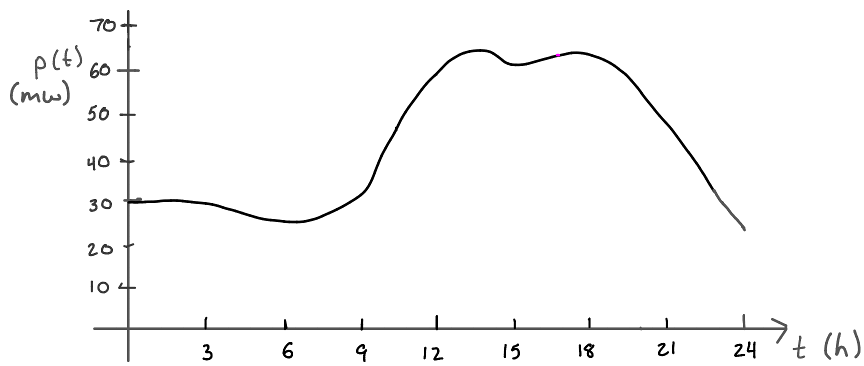


Figure 4.3. Power demand in a town over a 24-hour period

This graph is analogous to the graphs on the left-hand sides of Figures 4.1 and 4.2. Of course, the situation in Figure 4.3 is more complicated, in that the power is neither constant, nor constant "in steps;" instead, $p(t)$ is continuously varying with t. Still we wish to answer, at least approximately, the same kinds of questions as were posed in Parts A and B of this section.

Example 4.1.5. What, at least approximately, is the total amount of energy consumed by the above town, over the given 24-hour period?

Solution. They key is to break the interval [0,24] into smaller subintervals, such that, on each of these subintervals, the power function $p(t)$ does not vary "too much." On each of these intervals, we can then approximate energy consumption using (4.1.1). Adding together all of the energy consumption numbers obtained in this way, we can deduce an estimate for the total, cumulative energy usage over the entire period.

To illustrate, consider the following picture, where we have divided the 24-hour period of Figure 4.3 into ten different subintervals, such that $p(t)$ does not vary too much on any one of these subintervals.

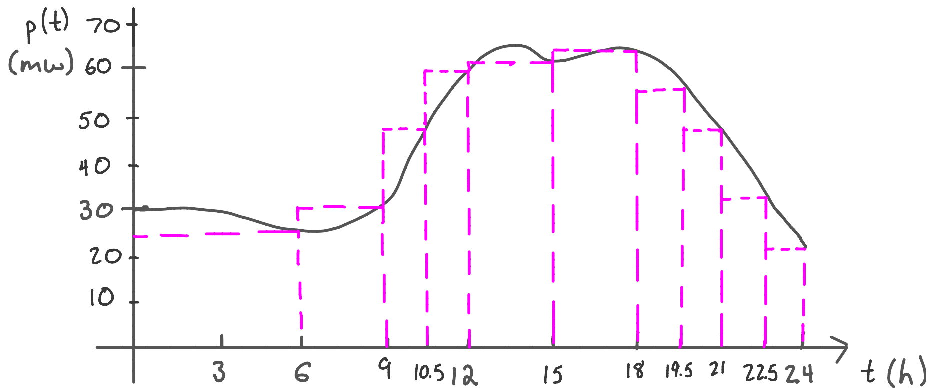


Figure 4.4. Breaking [0,24] into subintervals where $p(t)$ is "roughly constant"

Note that the subintervals have varying lengths. The idea here is this. In places where its graph is relatively flat (horizontal), $p(t)$ remains "roughly constant" for relatively long periods of time. So we can cover such parts of the graph with relatively long subintervals. Conversely, where its graph is steep, $p(t)$ changes rapidly. So, in such places, we need relatively short subintervals, to assure that $p(t)$ does not change too much on any one of those subintervals.

Since $p(t)$ is roughly constant on each subinterval, we can approximate $p(t)$, on any subinterval, by its value at *any point* – to be concrete, let's say the *right endpoint* – of that subinterval. (We could have chosen left endpoints, or midpoints, etc., just as well. We'll say more about this choice soon.) In other words, we can imagine that $p(t)$ is "constant in steps," and that the heights of the dashed, horizontal lines in Figure 4.3 give the values of $p(t)$ on the various steps.

Now the first of the subintervals in Figure 4.4 has length 6, and power value of about 24 (we read this value off of the graph as well as we can). Therefore, the energy consumption over this period is roughly 24 kw $\times$ 6 h $=$ 144 kwh. The second interval has length 3 and power value of about 30, so the energy consumption over this period is about 30 kw $\times$ 3 h $=$ 90 kwh. And so on up through the tenth subinterval.

The cumulative amount of energy consumed is the sum of energy values over all ten subintervals. Reading the appropriate power values off of the graph, then, we find that

$$\begin{aligned}
&\text{cumulative energy over the 24-hour period} \\
&\approx 24 \times 6 + 30 \times 3 + 48 \times 1.5 + 60 \times 1.5 + 62 \times 3 + 64 \times 3 \\
&+ 56 \times 1.5 + 48 \times 1.5 + 32 \times 1.5 + 22 \times 1.5 \\
&= 1011
\end{aligned} \tag{4.1.8}$$

megawatt-hours.

Note that the cumulative energy value obtained above is only an *estimate*. How might we get a better estimate? The answer is clear: start with a "step function" (that is, a function that is constant in steps) that approximates the power graph *more closely.* In principle, we can get as good an approximation as we might desire this way. We are limited only by the precision of the power graph itself.

For example, consider the following refinement of the subdivision shown in Figure 4.4. (Here, all subintervals have the same length, equal to 1.5 h.)

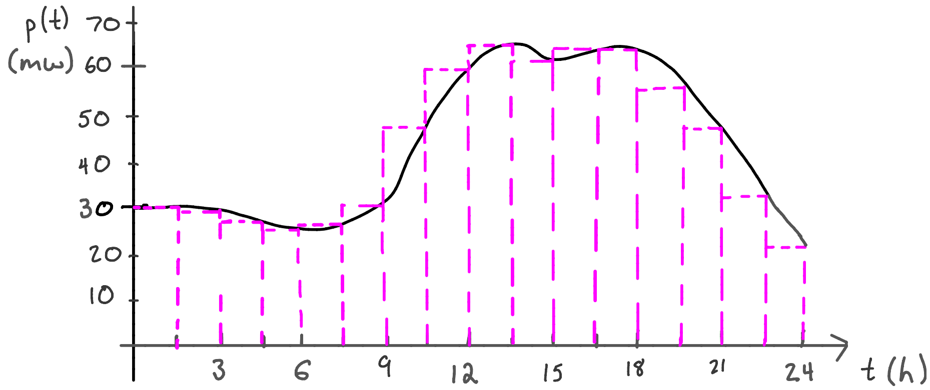


Figure 4.5. Breaking [0,24] into finer subintervals

From this picture, we might obtain the following approximation:

$$\begin{aligned}
&\text{cumulative energy over the 24-hour period}\\
&\approx 30\times1.5+29\times1.5+26\times1.5+24\times1.5+25\times1.5+30\times1.5+48\times1.5+60\times1.5\\
&+65\times1.5+61\times1.5+64\times1.5+63\times1.5+56\times1.5+48\times1.5+32\times1.5+22\times1.5\\
&=1024.5 \qquad (4.1.9)
\end{aligned}$$

megawatt-hours.

In summary, we determine the energy consumption of the town by a sequence of successive approximations. The steps in the sequence are listed in the box below.

1. **Approximate** the power demand by a step function.
2. **Estimate** energy consumption from this approximation.
3. **Improve** the energy estimate by choosing a new step function that follows power demand more closely.

Having approximated $p(t)$ by a step function as we did in Figure 4.4 above, we could now follow essentially the method of Example 4.1.4 to find an approximation to $E(T)$, the cumulative amount of energy consumed in our town between the given starting point and a point T hours later (up to $T = 24$). (Using Figure 4.5 instead, we could get an even better approximation to $E(T)$.) This approximation would be linear in steps, like the graph of $E(T)$ on the right-hand side of Figure 4.2, because our approximation to $p(t)$ is constant in steps.

Obtaining such an approximation would be somewhat arduous, because we have ten (or, in the case of Figure 4.5, sixteen) individual steps to consider. A simpler example is presented in Exercise 4 below.

In the following subsection, we consider $E(T)$ from the perspective of initial value problems. This perspective can sometimes allow us to deduce $E(T)$ from $p(T)$ in a more direct (and exact) way.

Part D: Accumulated energy consumption

In a general situation like that of Figure 4.3 above, energy is being consumed continuously, but at a varying rate, over the entire day. In such a situation, can we determine how much energy has been used through the first T hours of the day?

As above, we'll denote this quantity by $E(T)$, which we'll call the **energy accumulation function**. For example, we already have the estimate $E(24) = 1011$ mwh (or 1024.5 mwh) from formula (4.1.8) (or (4.1.9)) above.

To better understand $E(T)$ for more general values of T, choose any one of the subintervals from Example 4.1.5 above. Let's denote the energy consumed over just this subinterval by ΔE. Recall that, to approximate ΔE, we multiplied the elapsed time (the length of the subinterval) by the height of the power function at a point, call it t, in the subinterval. So, if we call the elapsed time Δt (as usual), then we have

$$\Delta E \approx p(t)\Delta t.$$

Dividing by Δt gives

$$\frac{\Delta E}{\Delta t} \approx p(t). \tag{4.1.10}$$

Now consider what happens to (4.1.10) as Δt shrinks to zero around the point t. The left-hand side, $\Delta E/\Delta t$, becomes the *derivative* dE/dt, by definition of the derivative. The "$\approx$" should become "$=$," since we expect better and better approximations as over narrow and narrower subintervals. So in the limit as $\Delta t \to 0$, (4.1.10) yields

$$\frac{dE}{dt} = p(t).$$

Note that, apart from notation, this is the same conclusion as was reached in equation (4.1.2), for the constant power situation, and in equation (4.1.7), for the case of power varying in steps.

In words, the differential equation $dE/dt = p$ says that *power is the rate at which energy is consumed*. In purely mathematical terms:

> **The energy accumulation function $E(t)$**
> **is a solution to the differential equation $dE/dt = p(t)$,**
> **where $p(t)$ is the power function.**

Relationship between energy and power

The arguments used to arrive at the above conclusion were a bit informal, but they can be made precise, so that the conclusion is in fact valid, as long as the power function in question is not too pathological.

In fact, it can be shown that $E(t)$ is **the** (only) solution to the **initial value problem**

$$\frac{dE}{dt} = p(t), \qquad E(0) = 0.$$

Note the import of this statement: *if* we can solve a certain initial value problem, then we can deduce the energy function from the power function.

In much the same way, cumulative effects of a great variety of phenomena can be understood in terms of initial value problems. We'll investigate such other instances, and the issue of accumulation in general, in the following sections.

To conclude the present section, we wish to make a geometric observation. Specifically: return to Figure 4.4 (or Figure 4.5), and consider any one of the subintervals there. Observe that we have the approximation

$$\begin{aligned}
&\text{energy consumption over the subinterval}\\
&\approx \text{ length of subinterval } \times \text{power reading at the right endpoint of the interval}\\
&= \text{ baselength of the rectangle drawn over the subinterval}\\
&\times \text{ height of the rectangle drawn over the subinterval}\\
&= \textit{area} \text{ of the rectangle drawn over the subinterval.}
\end{aligned}$$

So there's a relationship between energy and *area under a power curve.* This relationship exemplifies a more general connection between "accumulation" of certain functions and areas under the graphs of those functions. We'll be more explicit about this relationship in the next section.

Exercises

1. On Monday evening, a 1500 watt space heater is left on from 7 until 11 pm. How many kilowatt-hours of electricity does it consume?

2. Suppose a space heater has settings for 500, 1,000, and 1,500 watts.

(a) On Tuesday, we put this space heater on the 1,000 watt setting from 6 to 8 pm, then switch it to 1,500 watts from 8 till 11 pm, and then switch it to the 500 watt setting through the night until 8 am, Wednesday. How much energy is consumed, in kwh, by this heater over this $2 + 3 + 9 = 14$ hour period?

(b) Sketch the graphs of power demand $p(T)$ and accumulated energy consumption $E(T)$ for the space heater from Tuesday evening to Wednesday morning. Determine whether $E'(T) = p(T)$ in this case.

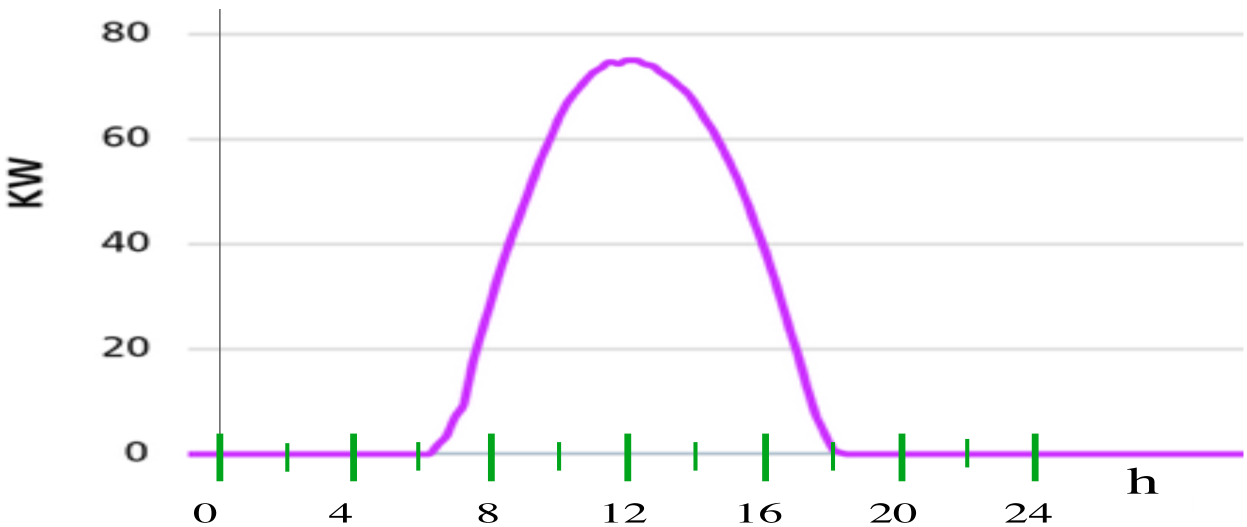
3. If energy is consumed (by an electronic device, a household, a town, etc.) over a period of time, then we define the **average power demand** over this period by:

$$\text{average power demand} = \frac{\text{energy consumption}}{\text{elapsed time}}.$$

(a) What is the average power demand of the space heater in Exercise 2(a) above, over the 14-hour period in question? What are your units?

(b) Figures 4.4 and 4.5 above depict the power demand of a town over a 24-hour period. Give two different estimates of the average power demand of the town during that period. Explain how you got your estimates.

4. The following graph depicts power *generated* (that is, power "into the grid") from solar cells, over the course of a day, at an elementary school in Boulder, Colorado. (Time $t = 0$, on the horizontal axis, corresponds to midnight, September 20, 2017.)



(a) Make a copy of the above graph, either electronically or by hand. On your copy, dash in a rectangle over each of the intervals [0,2], [2,4], [4,6],..., [22,24]. The height of the rectangle over each interval should be the value of the power function at the *right* endpoint of that interval. (Some of your rectangles will have height zero.)

(b) Let $E(T)$ denote cumulative energy generated by the above solar cells, from time 0 to time T. Use your above rectangles to estimate $E(2)$, $E(4)$, $E(6)$, ..., $E(24)$. (Careful: $E(T)$ measures *cumulative* energy. So, for example, calculation of $E(10)$ will involve the rectangles over [0,2], [2,4], [4,6], [6,8], and [8,10], not just the rectangle over [8,10].) What are the units for $E(T)$?

(c) On a separate graph, with time on the horizontal axis and energy on the vertical axis, plot the points $(0,E(0))$, $(2,E(2))$, $(4,E(4))$, ..., $(24,E(24))$. (Use your estimates from part (b) of this exercise.) Connect the dots with line segments, to sketch a very rough approximation to $E(T)$.

(d) What would you do to find a *better* approximation to $E(T)$? (You don't actually have to give a better approximation; just describe how you would.)

(e) What does the graph of $E'(T)$, the derivative of $E(T)$, look like? Hint: you don't need to draw any graphs to answer. In fact, by Part D of Section 4.1 above, you've already seen a graph of $E'(T)$. Where?