3.3 Inverse functions and the arctangent function

Much of what we have said about the natural exponential and logarithm functions carries over directly to *any* pair of inverse functions. To explain this, we should first say precisely what it means for two functions f and g to be inverses of each other.

Definition 3.3.1. Two functions f and g are **inverses** if

$$\text{(i) } f(g(b)) = b, \quad \text{and} \quad \text{(ii) } g(f(a)) = a, \tag{3.3.1}$$

for every b in the domain of g and every a in the domain of f.

For example: as previously noted, the functions $f(x) = e^x$ and $g(x) = \ln(x)$ are inverses of each other, by equations (3.2.5) and (3.2.6).

We mentioned in Section 3.5 that the functions $f(x) = e^x$ and $g(x) = \ln(x)$ are "flips," or reflections, of each other about the line $y = x$. This is a general property of pairs of inverse functions. That is: suppose we have *any* two functions $f(x)$ and $g(x)$ that are inverses of each other. Of course, the function $f(x)$ takes a number a in its domain to $f(a)$. But by equation (3.3.1)(ii), the function $g(x)$ then takes $f(a)$ back to a. Schematically, we have this picture:

$$a \longrightarrow \boxed{\begin{array}{c}\text{the function}\\ y = f(x)\end{array}} \longrightarrow f(a)$$

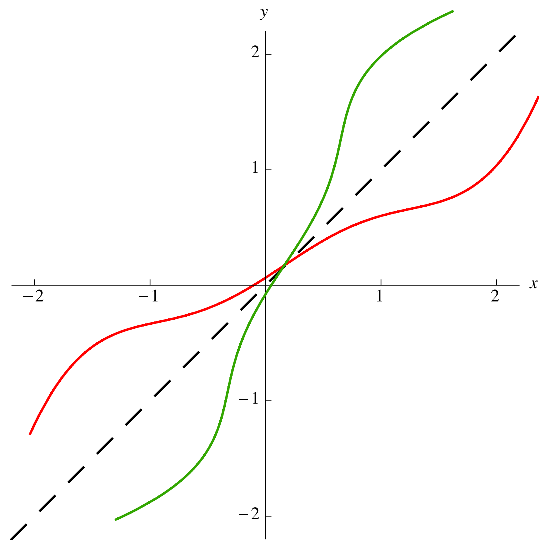
$$f(a) \longrightarrow \boxed{\begin{array}{c}\text{the function}\\ y = g(x)\end{array}} \longrightarrow a$$

In other words: the function $y = g(x)$ takes the function $y = f(x)$, and **reverses** the roles of input and output.

As noted earlier (in discussing exponential and logarithmic functions), swapping input with output amounts to swapping the horizontal with the vertical. And again, this is the same as *reflecting* everything about the line $y = x$. CONCLUSION:

If f and g are inverse functions of each other, then the graph of $y = g(x)$ is the graph of $y = f(x)$, reflected about the line $y = x$.

Geometrical relationship between inverse functions



Functions f (in red) and g (in green) that are inverse to each other. The graph of $y = g(x)$ is the graph of $y = f(x)$, reflected about the line $y = x$ (dashed).

Let $f(x)$ be a function that has an inverse function $g(x)$. We claim that $f(x)$ *must satisfy the horizontal line test*, meaning no horizontal line can intersect the graph of $f(x)$ more than once. Why is this true? Well, suppose there *were* a horizontal line $y = b$ intersecting the graph of $f(x)$ more than once. Then, since the graph of $g(x)$ is just that of $f(x)$ with the horizontal and vertical directions swapped, we would find that the *vertical* line $x = b$ intersects the graph of $g(x)$ more than once. But this is impossible, because we've assumed that $g(x)$ is a function, and functions must satisfy the vertical line test. (That is, no vertical line can intersect the graph of a function more than once.)

This motivates the following.

Definition 3.3.2. We say that a function f is **one-to-one**, usually written as **1–1**, if its graph satisfies the *horizontal line test:* no horizontal line intersects the graph more than once.

Here's another way of thinking about 1–1 functions. To say that no horizontal line intersects a graph more than once is to say that no y-value on the graph can come from two different x-values. So: to say that a function $y = f(x)$ is 1–1 is to say that, if inputs x_1 and x_2 are unequal, then the outputs $y_1 = f(x_1)$ and $y_2 = f(x_2)$ must be unequal as well.

We have seen that only functions that are one-to-one can have inverses. This means that to establish inverses for some functions, we will need to restrict their domains to regions where they are one-to-one. Let's consider some examples.

Example 3.3.1. Suppose $f(x) = x^2$ and $g(x) = \sqrt{x}$. The squaring function $f(x)$ is not invertible on its natural domain because it is not one-to-one there. (It's clear from the graph that $f(x) = x^2$ does not satisfy the horizontal line test. For example, the line $y = 9$ intersects this graph more than once. Or to put it another way, $f(-3) = f(3)$ even though $-3 \neq 3$.)

However, the squaring function $f(x) = x^2$ *is* invertible if we restrict its domain to non-negative real numbers. Then

$$f(g(b)) = \left(\sqrt{b}\right)^2 = b \ \text{(for } \ b \geq 0)$$
$$\text{and } g(f(a)) = \sqrt{a^2} = a \ \text{(for } \ a \geq 0).$$

The second of these statements – particularly the equation $\sqrt{a^2} = a$ – is only true on our restricted domain of f. It fails if we allow a to be negative – for example, $\sqrt{(-3)^2} = \sqrt{9} = 3 \neq -3$.

We already know how to find the derivative of the square root function. But let's compute this derivative again (assuming we know the derivative of the squaring function), to further illustrate how the derivative of an inverse function is related to that of the original function itself.

For our above functions $f(x) = x^2$ and $g(x) = \sqrt{x}$ we have, for appropriate values of x,

$$f(g(x)) = x.$$

Differentiate both sides: the derivative of the right-hand side is just 1, while the derivative of the left-hand side is given by the chain rule. We get

$$f'(g(x))g'(x) = 1$$

or, solving for $g'(x)$,

$$g'(x) = \frac{1}{f'(g(x))}. \tag{3.3.2}$$

But we can simplify the right-hand side of (3.3.2): since $f(x) = x^2$, we have $f'(x) = 2x$, so

$$f'(g(x)) = 2g(x) = 2\sqrt{x}.$$

So (3.3.2) gives $g'(x) = 1/(2\sqrt{x})$ or, recalling that $g(x)$ is the square root function,

$$\frac{d}{dx}\left[\sqrt{x}\,\right] = \frac{1}{2\sqrt{x}},$$

agreeing with earlier results.

Note that we could have restricted the domain of f in another way to make it one-to-one: we could have taken its domain to consist of non-positive real numbers $x \leq 0$, instead of non-negative real numbers $x \geq 0$. Now the function $g(x) = \sqrt{x}$ is no longer the inverse of this restricted f. For instance, $g(f(-3)) = g(9) = 3 \neq -3$. What would the inverse of f be in this case? (See Exercise 3 below.)

Example 3.3.2. Suppose $y = \tan(x)$. Note that this does not define a one-to-one function on the natural domain of the tangent function. For example, $\tan(\pi/4)$ and $\tan(5\pi/4)$ are both equal to 1, even though $\pi/4 \neq 5\pi/4$. (In fact, the the horizontal line $y = 1$ intersects the graph of $y = \tan(x)$ infinitely often, since $\tan(k\pi + \pi/4) = 1$ for every integer k.) So we need to restrict our domain in order for our tangent function to have an inverse.

To this end, let's consider the function $f(x) = \tan(x)$ with domain $-\pi/2 < x < \pi/2$.

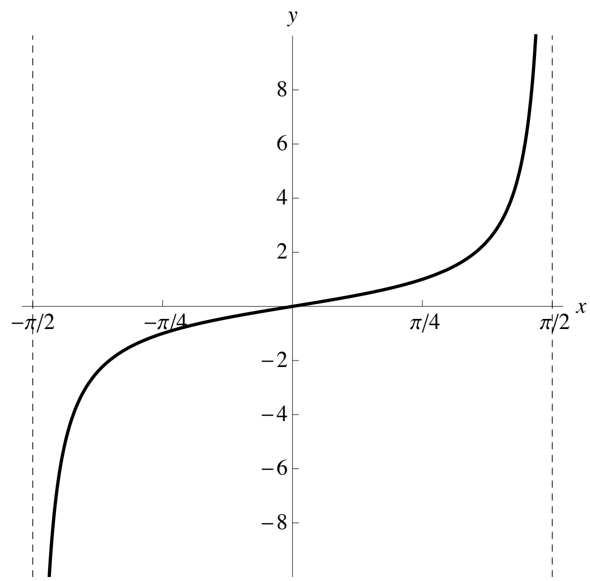


Figure 3.10. The graph of $f(x) = \tan(x)$ on $(-\pi/2,\pi/2)$

As is clear from the graph, this function *is* 1–1 on the indicated domain. So it has an inverse function $g(x)$ there. This inverse function is called the *arctangent* function, denoted $\arctan(x)$. Since $f(x) = \tan(x)$ has domain $(-\pi/2,\pi/2)$ and range $(-\infty,\infty)$, we find that $g(x) = \arctan(x)$ has domain $(-\infty,\infty)$ and range $(-\pi/2,\pi/2)$. The graph of $g(x)$ looks like this.

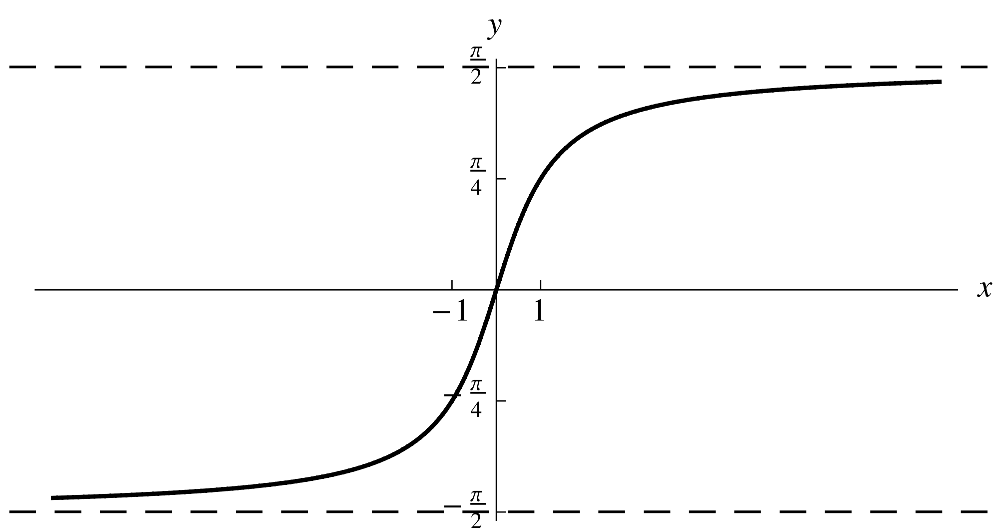


Figure 3.10. The graph of $g(x) = \arctan(x)$

To find the derivative of the arctangent function, we proceed in the usual way. Specifically: we start with the fact that

$$\tan(\arctan(x)) = x.$$

We differentiate both sides; on the left-hand side, we use the chain rule, and the fact that the derivative of the tangent function is the square of the secant function. We get

$$\sec^2(\arctan(x))\frac{d}{dx}[\arctan(x)] = 1$$

or, dividing both sides by $\sec^2(\arctan(x))$,

$$\frac{d}{dx}[\arctan(x)] = \frac{1}{\sec^2(\arctan(x))}. \tag{3.3.3}$$

We can simplify the right-hand side using the trigonometric identity

$$\sec^2(\theta) = 1 + \tan^2(\theta).$$

Note that this identity implies

$$\sec^2(\arctan(x)) = 1 + \tan^2(\arctan(x)) = 1 + \big(\tan(\arctan(x))\big)^2 = 1 + x^2, \tag{3.3.4}$$

the last step because, again, $\tan(\arctan(x)) = x$. Putting (3.3.4) into (3.3.3) gives us our ultimate result:

$$\boxed{\frac{d}{dx}[\arctan(x)] = \frac{1}{1+x^2}}$$

Derivative of the arctangent function

Example 3.3.3. Find (i) $\frac{d}{dx}[\arctan(2+x^5)]$ and (ii) $\frac{d}{dr}[2+\arctan^5(r)]$.

Solution. (i) $\frac{d}{dx}[\arctan(2+x^5)] = \frac{1}{1+(2+x^5)^2}\frac{d}{dx}[2+x^5] = \frac{5x^4}{1+(2+x^5)^2}$.

(ii) $\frac{d}{dr}[2+\arctan^5(r)] = \frac{d}{dr}[2+(\arctan(r))^5] = 0 + 5(\arctan(r))^4\frac{d}{dx}[\arctan(r)] = \frac{5\arctan^4(r)}{1+r^2}$.

To conclude this section we observe that, if f and g are inverse functions, then the chain rule gives us a method for finding the derivative $g'(x)$, given knowledge of $f'(x)$. We've seen this method employed in the context of exponential and logarithmic functions (see the subsection "The derivative of the logarithm function" in Section 3.5); in the context of squares and square roots (see Example 3.3.1); and in the context of the tangent and arctangent functions (see Example 3.3.2). The strategy employed, in each case, was as follows:

Step 1. Begin with the formula $f(g(x)) = x$, where f is the function whose derivative is already known.
Step 2. Differentiate to get $f'(g(x))g'(x) = 1$.
Step 3. Divide by $g'(x)$ to get $g'(x) = \frac{1}{f'(g(x))}$.
Step 4. Simplify the right-hand side if possible. This simplification will often use the fact that $f(g(x)) = x$.

Strategy for differentiating inverse functions

Step 4 is often the "tricky" step. In the case of exponents and logarithms, or squares and square roots, this step was fairly straightforward. But in the case of tangents and arctangents, we needed to make the non-obvious observation that $\sec^2(\theta) = 1 + \tan^2(\theta)$. This allowed us to obtain an expression involving $\tan(\arctan(x))$, which simplified because the tangent and arctangent functions are inverse to each other.

Finally: since an inverse to a function $y = f(x)$ (when an inverse exists) is obtained by interchanging the roles of x and y (that is, of input and output), we can sometimes find an inverse function, algebraically, by making this interchange – that is, by writing $x = f(y)$ – and solving for y.

Example 3.3.4. To find the inverse of the function

$$y = \frac{3-x}{2+x},$$

we interchange x with y:

$$x = \frac{3-y}{2+y},$$

and then solve for y:

$$\begin{aligned} x(2+y) &= 3-y \\ 2x + xy &= 3-y \\ xy + y &= 3-2x \\ y(x+1) &= 3-2x \\ y &= \frac{3-2x}{x+1}. \end{aligned}$$

As a check on our work in the above example, we show that the functions $f(x) = \dfrac{3-x}{2+x}$ and $g(x) = \dfrac{3-2x}{x+1}$ truly are inverses, as follows:

$$\begin{aligned} f(g(x)) = f\left(\frac{3-2x}{x+1}\right) &= \frac{3-(3-2x)/(x+1)}{2+(3-2x)/(x+1)} \\ &= \frac{3(x+1)-(3-2x)}{2(x+1)+(3-2x)} = \frac{3x+3-3+2x}{2x+2+3-2x} \\ &= \frac{5x}{5} = x, \end{aligned}$$

as required. (For the third "=" in this computation, we multiplied top and bottom by $x+1$, to simplify.)

Note that, once we've shown that $f(g(x)) = x$, we don't need to also check that $g(f(x)) = x$. Why? Essentially because, if $f(g(x)) = x$ for all x in the domain of g, then $f(x)$ is the reflection of $g(x)$ about the line $y = x$. And geometrically, this is the same as $g(x)$ being the reflection of $f(x)$ about the line $y = x$, which means $g(f(x)) = x$ for all x in the domain of f.

Example 3.3.5. Find the inverse g to the function $f(x) = x^2 - 4$ on the domain $x \geq 0$. What is the domain of g?

Solution. We write $y = x^2 - 4$, interchange x and y, and solve for y:

$$\begin{aligned} x &= y^2 - 4 \\ y^2 &= x + 4 \\ y &= \pm\sqrt{x+4} \end{aligned}$$

We *must* choose the plus sign, because we have stipulated that $f(x)$ have only nonnegative numbers in its domain, so its inverse must have only nonnegative numbers in its range.

So our inverse function is $g(x) = \sqrt{x+4}$. The domain of $g(x)$ is the set of all $x \geq -4$, since this is the range of $f(x)$.

Exercises

1. Find:

(a) $\frac{d}{dx}[3\arctan 4x]$

(b) $\frac{d}{dx}[e^{\arctan(x)}]$

(c) $\frac{d}{dx}\left[\arctan\left(\frac{1}{x}\right)\right]$

(d) $\frac{d}{dq}[(1+q^2)\arctan(q) - q]$

(e) $\frac{d}{dy}[\arctan^3(y\ln(y))]$

(f) $\frac{d}{dy}\left[\frac{1}{1+\arctan(y)}\right]$

2. Consider the function $f(x) = 3x^2 - 5$ on the domain $x \geq 0$. What is the inverse function to $f(x)$ on this domain? What is the domain of this inverse function?

3. Consider the function $f(x) = x^2$ on the domain $x \leq 0$. What is the inverse function to $f(x)$ on this domain? What is the domain of this inverse function?

4. Show that $f(x) = 1 - x$ equals its own inverse. What are the domain and range of f?

5. Show that $f(x) = 1/x$ equals its own inverse. What are the domain and range of f?

6. Let n be a positive integer. and let $f(x) = x^n$. What is an inverse of f? How do we need to restrict the domain of f for it to have an inverse? Caution: the answer depends on n.

7. What is the inverse g of the function $f(x) = 1 - 3x$?

8. What is the inverse g of the function $f(x) = \frac{1-3x}{2x+5}$?

9. What is the inverse of $f(x) = \dfrac{x^2}{2} + 5$ on the domain $x \geq 0$?

10. Use the strategy in the box on page 155 and the fact that $d[x^3]/dx = 3x^2$ to derive the formula for the derivative of $\sqrt[3]{x}$. (Pretend you don't already know how to differentiate $\sqrt[3]{x}$.) See Example 3.3.1 for a similar problem.

11. (a) Use a computer graphing utility to graph $y = \sin(x)$ on the domain $-\pi/2 \leq x \leq \pi/2$.

(b) Explain in words how you can tell from the graph that $y = \sin(x)$ has an inverse function on $[-\pi/2,\pi/2]$.

(c) Denote the inverse function from part (b) of this exercise by $\arcsin(x)$. What is the domain of $\arcsin(x)$?

(d) Use Steps 1–3 of the strategy on page 155 to show that

$$\frac{d}{dx}[\arcsin(x)] = \frac{1}{\cos(\arcsin(x))}.$$

(See also Example 3.3.2 for a similar problem.)

(e) Use the fact that $\cos(x) = \sqrt{1 - \sin^2(x)}$ on the domain $[-\pi/2,\pi/2]$ to conclude that

$$\frac{d}{dx}[\arcsin(x)] = \frac{1}{\sqrt{1 - x^2}}.$$