3.2 The natural logarithm function

We further examine some exponential growth/decay contexts considered in Section 3.1, and develop some tools to solve certain kinds of problems that arise in those contexts.

Solving the equation $e^a = b$

In Section 3.1, we solved exponential growth and decay problems problems of the "how large/how much" variety. That is: using formulas like

$$P = P_0 e^{kt} \quad \text{and} \quad R = R_0 e^{-kt}, \tag{3.2.1}$$

we were able to answer some questions like "how large is this population after that many years," or "how much of this radioactive substance is left after that many hours?" Answers to such problems entail simply putting the appropriate value of t into the appropriate formula (3.2.1) for the exponentially growing/decaying substance.

What we have not yet considered are answers to "how long" questions, in exponential growth/decay situations. That is, how do we solve equations like (3.2.1) for t, given values of the other variables and constants involved?

For example, consider the following.

Example 3.2.1. Suppose a population, initially comprising 100,000 persons, is growing at the per capita rate of $k = 3$ births per thousand persons per year.

(a) Write down an initial value problem modeling this situation.

(b) How large will this population be 37 years from now?

(c) How long will it take the population to double?

Solution. (a) Denoting time in months by t, and population in individuals by $P(t)$, we have the initial value problem

$$P'(t) = 0.003P(t); \qquad P(0) = 100{,}000.$$

(b) Using the results of Section 3.1, we know that the solution to the problem is the exponential function

$$P(t) = 100{,}000\, e^{0.003t}.$$

The size of the population 37 years from now will therefore be

$$\begin{aligned} P(37) &= 100{,}000\, e^{0.003(37)} \\ &= 100{,}000 \times 1.117395 \\ &\approx 111{,}740 \text{ people.} \end{aligned}$$

(c) To find out by when the population will double, we want to find a value of t such that $P(t) = 200{,}000$. In other words, we need to solve for t in the equation

$$100{,}000\, e^{0.003\,t} = 200{,}000.$$

Dividing both sides by 100,000, we have

$$e^{0.003t} = 2. \tag{3.2.2}$$

At the moment, we don't have the technology to solve (3.2.2), because one side is in exponential form, but the other isn't.

To solve (3.2.2), we need answer this question: how do we "get the $0.003t$ out of the exponent"? More generally, what we're asking is: if we know what e to the something is, then how do we solve for that something? Or, in more formal mathematical language: if we know that $e^a = b$, then how do we solve for a?

First of all, recall that e^a is always positive. This means that $e^a = b$ has *no solution* if $b \leq 0$. So we restrict our attention to positive numbers b.

For such b, the answer to the question "how do we solve $e^a = b$ for a?" lies, perhaps surprisingly, in a quantity that we've seen before, in the context of the derivative of the function b^x. Namely, we have the following:

The solution to the equation $e^a = b$, for $b > 0$, is $a = \ln(b)$

The relationship between the natural exponential and the natural logarithm function

We will use the above relationship to solve Example 3.2.1(c). We first wish to prove this relationship.

The proof, while short, relies on a somewhat non-intuitive trick: we let

$$f(x) = b^x \quad \text{where} \quad b = e^a,$$

and consider $f'(0)$, the derivative of $f(x)$ at $x = 0$, in two different ways.

On the one hand, we have $f'(x) = \ln(b)b^x$; plugging in $x = 0$ gives

$$f'(0) = \ln(b) \cdot b^0 = \ln(b) \cdot 1 = \ln(b). \tag{3.2.3}$$

On the other hand, we have $f(x) = b^x = (e^a)^x = e^{ax}$, so $f'(x) = ae^{ax}$, so

$$f'(0) = ae^{a\cdot 0} = ae^0 = a \cdot 1 = a. \tag{3.2.4}$$

As the left-hand sides of equations (3.2.3) and (3.2.4) agree, the right-hand sides must agree as well. So, under our assumption that $b = e^a$, we have found that $a = \ln(b)$, as claimed.

There is a technical point here that needs to be made. We have shown that $a = \ln(b)$ is **a** solution to $e^a = b$; how do we know it's **the** (only) solution? After all: the equation $a^2 = 9$, for example, has *two* solutions: $a = 3$ and $a = -3$. And the equation $\sin(a) = 0$ has infinitely many: $a = 0, \pm\pi, \pm 2\pi, \pm 3\pi, \ldots$. What's special about the exponential function that guarantees only **one** solution to the equation $e^b = a$?

The answer is: *the function $y = e^x$ satisfies the horizontal line test.* What this means is: each horizontal line intersects the graph of $y = e^x$ *at most* once. That this is true is clear from Figure 3.1 above.

A horizontal line has equation $y = b$, for some real number b. So: to say that every horizontal line intersects the graph of $y = e^x$ at most once is to say that, for any real number b, $y = b$ and $y = e^x$ intersect at most once. This means that $e^x = b$ has at most one solution.

Note that "at most once" does not guarantee "at least once." Indeed, as is seen from Figure 3.1 above, the graph of $y = e^x$ does not intersect the horizontal line $y = b$ **at all** if $b \leq 0$. So, as already noted, $e^x = b$ has no solution if $b \leq 0$.

What all of the above amounts to is this: the two statements

$$e^a = b \quad \text{and} \quad \ln(b) = a$$

express exactly the same relation between the quantities a and b. Note that, if we plug the second of these equations in to the first, then we get

$$e^{\ln(b)} = b \quad \text{(for any positive number } b\text{)}, \tag{3.2.5}$$

while, if we plug the first into the second, we get

$$\ln(e^a) = a \quad \text{(for any real number } a\text{)}. \tag{3.2.6}$$

The equations (3.2.5) and (3.2.6) express the fact that **the natural logarithm and natural exponential functions are inverses of each other**. Here, the term "inverse" is used not to mean that these functions are reciprocals of each other (they're not!), but instead that they "undo" each other. That is: if you start with a positive number b, take its natural logarithm, and then raise e to the result, then (by (3.2.5)) you get back what you started with. Similarly: if you start with a real number a, raise e to that number, and then take the natural logarithm of the result, then (by (3.2.6)) you get back what you started with.

Let's return to Example 3.2.1(c). There we encountered the question that led to the introduction of the logarithm function in the first place, namely: how do we solve the equation

$$e^{0.003t} = 2?$$

We now have a way. Specifically, we take the natural logarithm on both sides of this equation, to get

$$\ln(e^{0.003t}) = \ln(2).$$

But by equation (3.2.6) above, $\ln(e^{0.003t}) = 0.003t$. So we get

$$0.003t = \ln(2)$$

or, dividing by t and using a calculator to evaluate $\ln(2)$,

$$t = \frac{\ln(2)}{0.003} = 231.049....$$

So, in answer to the original question: it takes a little over 231 years for this population to double.

Many pairs of functions that share a key on a calculator – sine and arscsine, squareroot and squaring – are inverses of each other. There are even functions (at least one can be found on most calculators) that are their own inverses – apply such a function to any number, then apply this same function to the result, and you're back at the original number. What functions do this? We will say more about inverse functions in the next section.

Properties of the natural logarithm function

Algebraic properties. The inverse relationship between exponents and logarithms – that is, the fact that they "undo" each other – allows us to translate each property of the exponential function into a corresponding statement about the logarithm function. We list the major pairs of properties below.

exponential version	**logarithmic version**
$e^0 = 1$	$\ln(1) = 0$
$e^{a+b} = e^a \cdot e^b$	$\ln(mn) = \ln(m) + \ln(n) \quad (m,n > 0)$
$e^{a-b} = e^a / e^b$	$\ln(m/n) = \ln(m) - \ln(n) \quad (m,n > 0)$
$(e^a)^s = e^{as}$	$\ln(m^s) = s \cdot \ln(m) \quad (m > 0)$
range of e^x is all positive reals	domain of $\ln(x)$ is all positive reals
domain of e^x is all real numbers	range of $\ln(x)$ is all real numbers
$e^x \to 0$ as $x \to -\infty$	$\ln(x) \to -\infty$ as $x \to 0$
e^x goes to $+\infty$ than x^n, for any $n > 0$	$\ln(x)$ goes to $+\infty$ slower than $x^{1/n}$, for any $n > 0$

For each of these pairs of properties, we can use the exponential property and the inverse relationship between exponential and logarithmic functions to establish the corresponding logarithmic property. As an example, we will establish the second property. You should be able to demonstrate the others.

Proof of the second property. We wish to use the property $e^{a+b} = e^a e^b$ to deduce the property $\ln(mn) = \ln(m) + \ln(n)$. To do this note that, since we are assuming m and n to be positive numbers, there must be real numbers a and b such that $e^a = m$ and $e^b = n$. But then

$$\ln(mn) = \ln(e^a e^b) = \ln(e^{a+b}) = a + b = \ln(m) + \ln(n),$$

and our proof is complete.

Example 3.2.2. Simplify

$$\ln\left(\frac{a^{10}e^{x^2}}{b^{\sin(x)}}\right).$$

Solution. By some of the properties listed above (which ones?),

$$\begin{aligned}\ln\left(\frac{a^{10}e^{x^2}}{b^{\sin(x)}}\right) &= \ln\left(a^{10}e^{x^2}\right) - \ln\left(b^{\sin(x)}\right)\\ &= \ln\left(a^{10}\right) + \ln\left(e^{x^2}\right) - \ln\left(b^{\sin(x)}\right)\\ &= 10\ln(a) + x^2 - \sin(x)\ln(b).\end{aligned}$$

Geometric properties. We now consider the function $y = \ln(x)$ geometrically: what does the graph of this function look like?

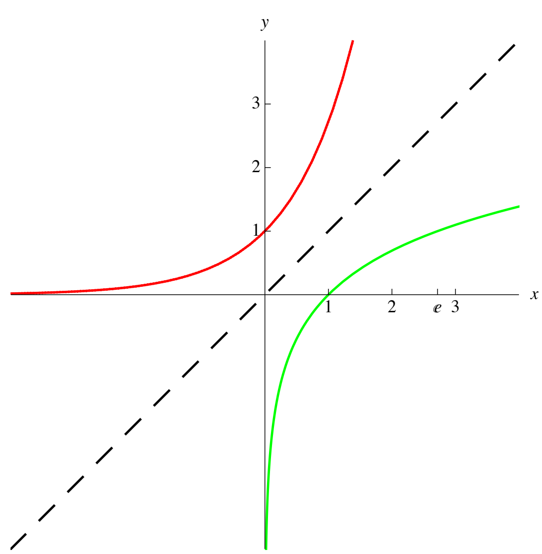
To answer, we begin by noting that (obviously) the function $y = e^x$ takes an input a to an output e^a. But now observe that, by (3.2.6), $\ln(e^a) = a$, so that the function $y = \ln(x)$ takes an input e^a to an output a. Schematically, we have this picture:

$$a \longrightarrow \boxed{\begin{array}{c}\text{the function}\\ y = e^x\end{array}} \longrightarrow e^a$$

$$e^a \longrightarrow \boxed{\begin{array}{c}\text{the function}\\ y = \ln(x)\end{array}} \longrightarrow a$$

In other words: *the function* $y = \ln(x)$ *takes the function* $y = e^x$*, and* **reverses** *the roles of input and output.*

Geometrically, swapping input with output amounts to swapping the horizontal with the vertical. And as one can show, this is the same as *reflecting* everything about the line $y = x$. CONCLUSION: the graph of $y = \ln(x)$ **is** the graph of $y = e^x$, reflected about the line $y = x$!



The graph of $y = \ln(x)$ (in green) is the graph of $y = e^x$ (in red), reflected about the line $y = x$ (dashed)

The derivative of the logarithm function

Because the functions $y = e^x$ and $y = \ln(x)$ "undo" each other, our knowledge about the derivative of the first of these functions can be transformed into a formula for the derivative of the second.

Here's how. We begin with equation (3.2.5) (with x in place of b):

$$e^{\ln(x)} = x.$$

We differentiate both sides:

$$\frac{d}{dx}\left[e^{\ln(x)}\right] = \frac{d}{dx}[x]. \tag{3.2.7}$$

The right-hand side is simply equal to 1. The left-hand side equals $e^{\ln(x)} \cdot d[\ln(x)]/dx$, by the chain rule. So (3.2.7) gives

$$e^{\ln(x)} \frac{d}{dx}[\ln(x)] = 1$$

or, dividing both sides by $e^{\ln(x)}$,

$$\frac{d}{dx}[\ln(x)] = \frac{1}{e^{\ln(x)}}. \tag{3.2.8}$$

We've found a formula for the derivative of $\ln(x)$! And we can further simplify this formula by plugging into the right-hand side the fact that, again, $e^{\ln(x)} = x$. So (3.2.8) reads

$$\boxed{\frac{d}{dx}[\ln(x)] = \frac{1}{x}}$$

Derivative of the natural logarithm function

Example 3.2.3. Find:

(a) $\frac{d}{dx}[\cos(1 - \ln(x))]$; (b) $\frac{d}{dx}[\ln(1 - \cos(x))]$; (c) $\frac{d}{dx}\left[\ln\left(\frac{x^2}{\ln(x^2)}\right)\right]$.

Solution. (a) By the chain rule,

$$\begin{aligned}\frac{d}{dx}[\cos(1 - \ln(x))] &= -\sin(1 - \ln(x))\frac{d}{dx}[1 - \ln(x)] \\ &= -\sin(1 - \ln(x))\left(0 - \frac{1}{x}\right) = \frac{\sin(1 - \ln(x))}{x}.\end{aligned}$$

(b) Again by the chain rule,

$$\begin{aligned}\frac{d}{dx}[\ln(1 - \cos(x))] &= \frac{1}{1 - \cos(x)}\frac{d}{dx}[1 - \cos(x)] \\ &= \frac{1}{1 - \cos(x)} \cdot (0 + \sin(x)) = \frac{\sin(x)}{1 - \cos(x)}.\end{aligned}$$

(c) We could differentiate this directly, using the chain rule (multiple times) and the quotient rule. But it's easier to first simplify the function being differentiated, using properties of logarithms:

$$\begin{aligned}\ln\left(\frac{x^2}{\ln(x^2)}\right) &= \ln(x^2) - \ln(\ln(x^2)) \\ &= 2\ln(x) - \ln(2\ln(x)) = 2\ln(x) - \ln(2) - \ln(\ln(x)).\end{aligned}$$

Then

$$\begin{aligned}\frac{d}{dx}\left[\ln\left(\frac{x^2}{\ln(x^2)}\right)\right] &= \frac{d}{dx}[2\ln(x) - \ln(2) - \ln(\ln(x))] \\ &= \frac{2}{x} - 0 - \frac{1}{\ln(x)}\frac{d}{dx}[\ln(x)] = \frac{2}{x} - \frac{1}{x\ln(x)}.\end{aligned}$$

(Since $\ln(2)$ is just a constant, its derivative is zero.)

We have seen that the function $y = \ln(x)$ has the following properties:

$$\frac{dy}{dx} = \frac{1}{x}; \qquad y(1) = 0. \tag{3.2.9}$$

That: $y = \ln(x)$ solves the *initial value problem* (3.2.9). We can use this fact with Euler's method to compute values of $\ln(x)$. Applications of this idea can be found in the exercises.

Exponential growth and decay, revisited

With the logarithm function in hand, we are now able to perform more detailed, complete analyses of exponential growth or decay scenarios. In particular, we can now:

- Solve "how long" problems (like Example 3.2.1(c) above, which we solved on page 139);
- Evaluate certain parameters, like per capita growth rates and per unit decay rates, given certain additional information about particular values of the quantities in question.
- Evaluate certain related quantities, like "half-lives" (see Example 3.2.4 below) and "doubling times" (see, again, Example 3.2.1(c) and its solution, above).
- Express exponential growth or decay solutions in terms of other bases, which are sometimes more suggestive of the behavior in question.

Here are some examples to illustrate these ideas.

Example 3.2.4. In Exercise 14 of Section 1.5, we saw that radium 226 has per unit decay rate $k = 1/2337$ year^{-1}. Use this information to find the "half-life," call it τ, of radium 226, meaning the length of time τ that it takes for a given sample to reduce to half of its original mass.

Solution. As we've seen, the amount R of radium 226 present satisfies the differential equation $R' = -(1/2337)R$, if time is measured in years and R is in grams. By what we've seen about exponential decay, then, we know that a sample of R_0 grams of radium reduces to

$$R(t) = R_0 e^{-t/2337} \tag{3.2.10}$$

grams after t years.

We want to know what time value $t = \tau$ leave us with half of what we started with. That is, we want to solve $R(\tau) = (1/2)R_0$, or by (3.2.10),

$$R_0 e^{-\tau/2337} = \frac{1}{2} R_0.$$

Divide both sides by R_0 to get

$$e^{-\tau/2337} = \frac{1}{2}.$$

Now take natural logarithms of both sides to get

$$-\frac{\tau}{2337} = \ln\left(\frac{1}{2}\right).$$

Finally, solve for τ:

$$\tau = \frac{\ln(1/2)}{-2337} = 1619.88....$$

That is, the half-life of radium 226 is about 1620 years.

Note that, to solve the above example, we **did not** need to know how much radium we started with. We denoted this initial quantity by R_0 but, since we divided through by R_0 along the way, we ended up with a solution that did not depend on R_0. This is characteristic of exponential decay: *if a quantity decays exponentially, then the amount of time it takes for this quantity to halve, or to reduce to one-third of its original amount, or to reduce to p percent of its original amount, for any number p, does not depend on that original amount.* (Of course, it does depend on p: you'll be down to 75% of the original amount before you're left with only 10%, and so on.) Similarly for exponential growth: *if a quantity grows exponentially, then the amount of time it takes for this quantity to double, or triple, or grow to a factor of f times the original amount, for any number f, does not depend on that original amount.* (But it does depend on f.)

These claims are true essentially by the arguments used in solving Examples 3.2.1(c) and 3.2.4 above: ultimately, the original amount R_0 (or whatever it's called) gets "divided out" of the equation. For a more thorough argument, see the exercises below.

Example 3.2.5. A certain exponentially growing population triples every five years.

(a) What is the per capita growth rate k for this population?

(b) If $P_0 = 10$ (in millions), then (i) find an explicit formula for $P(t)$, with t in years; and (ii) how long does it take for the population to reach 250 million?

Solution. (a) If t is in years and $P(t)$ in millions of people, then

$$P(t) = P_0 e^{kt}, \tag{3.2.11}$$

where P_0 is also in millions of people.

We're told that $P(5) = 3P_0$ or, by (3.2.11),

$$P_0 e^{k \cdot 5} = 3P_0.$$

Divide by P_0:

$$e^{5k} = 3.$$

Take the natural logarithm on both sides:

$$5k = \ln(3).$$

Solve for k:

$$k = \frac{\ln(3)}{5} = 0.2197. \tag{3.2.12}$$

The per capita growth rate is $k = 0.2197$ year^{-1}.

(b) (i) By part (a),

$$P(t) = 10e^{0.2197t}$$

millions of people after t years. So: (ii) the population equals 250 million when

$$10e^{0.2197t} = 250.$$

We solve for t by dividing both sides by 10, taking natural logarithms of both sides, and then dividing both sides by 0.2197:

$$\begin{aligned} e^{0.2197t} &= \frac{250}{10} = 25 \\ 0.2197t &= \ln(25) \\ t &= \frac{\ln(25)}{0.2197} = 14.6512. \end{aligned}$$

It takes about 14.6512 years for the population to reach 25 million.

The above formula $P(t) = 10e^{0.2197t}$ is useful for calculations. But there is another way of writing this formula, which is more directly reflective of the way $P(t)$ grows, in this case.

Namely: we saw in (3.2.12) that $k = \ln(3)/5$. So, using properties of exponents and logarithms, we can rewrite the formula $P(t) = 10e^{kt}$ as follows:

$$P(t) = 10e^{kt} = 10e^{(\ln(3)/5)t} = 10e^{\ln(3)\cdot(t/5)} = 10(e^{\ln(3)})^{t/5} = 10 \cdot 3^{t/5}. \tag{3.2.13}$$

This new way of writing $P(t)$ makes it clear that $P(t)$ triples every five years, since the quantity $3^{t/5}$ on the right-hand side of (3.2.13) becomes three times as large when we increase t by 5. (That is, $3^{(t+5)/5} = 3 \cdot 3^{t/5}$, as you should verify.)

Another thing we should note about the above example is that the result agrees with the following "gut check." We are told that our population triples every five years. So after five years, our original population of 10 million has grown to 30 million. After *another* five years the population of 30 million triples, to 90 million. And after *another* five years, the population of 90 million triples, to 270 million. Thus, after a grand total of $5+5+5=15$ years, our initial 10 million individuals have become 270 million. This is entirely consistent with, and gives us more confidence in, our finding that, after 14.6512 years (a bit less than 15), our population has grown to 250 million (a bit less than 270 million). Or to put it another way: the result $P(14.6512) = 250$ (million) is pretty close to the result $P(15) = 270$ (million), so we feel more comfortable about the former result, even though the calculations there were just a bit messy.

Example 3.2.6. Salt dissolves in water at a rate proportional to the amount $S(t)$ of salt remaining at time t.

If 6 pounds of salt reduce to 5 pounds after one hour, how much salt remains after four hours?

Solution. If we measure t in hours (h) and $S(t)$ in pounds (lb), then we may model our situation by the initial value problem

$$S' = -kS; \qquad S(0) = 6.$$

We know that this problem has solution

$$S(t) = 6e^{-kt} \tag{3.2.14}$$

for some positive constant k.

We find k by substituting $S(1) = 5$ into (3.2.14); we get

$$5 = 6e^{-k\cdot 1}.$$

We solve for k:

$$\frac{5}{6} = e^{-k}$$

$$\ln\left(\frac{5}{6}\right) = -k$$

$$k = -\ln\left(\frac{5}{6}\right) = 0.182322 \text{ h}^{-1}. \tag{3.2.15}$$

So (3.2.14) reads

$$S(t) = 6e^{-0.182322t}.$$

Finally, we compute:

$$S(4) = 6e^{-0.182322\cdot 4} = 2.8935 \text{ lb}.$$

Another way of thinking through the above example is this. The equation $k = -\ln(5/6)$, from (3.2.15), gives us

$$S(t) = 6e^{-kt} = 6e^{-(-\ln(5/6))t} = 6(e^{\ln(5/6)})^t = 6\cdot(5/6)^t.$$

This latter equation reflects the fact that the amount of salt remaining reduces by a factor of $5/6$ every hour. So after one hour, we have $6\cdot(5/6) = 5$ lb left; after two hours, we have $6\cdot(5/6)^2$ lb; after three, we have $6\cdot(5/6)^3$ lb; after four, we have $6\cdot(5/6)^4 = 2.8935$ lb, as above.

Exercises

Part 1: Basic properties of the logarithm function

1. Determine the numerical value of each of the following. Do not use a calculator for these; just use properties of exponents and logarithms. Your answer to each of the parts of this exercise should be an integer or a fraction (that is, an integer divided by an integer).

(a) $(\ln(e))^5$	(b) $\ln(e^5)$	(c) $4^{\ln(e)}$	(d) $\ln(\sqrt{e})$
(e) $e^{\ln(2)}$	(f) $e^{3\ln(2)}$	(g) $(e^{\ln(2)})^3$	(h) $e^{2\ln(3)}$
(i) $e^{-\ln(2)}$	(j) $e^{-3\ln(2)}$	(k) $5e^{\ln(6/5)}$	(l) $e^{\ln(2)+\ln(3)}$
(m) $e^{\ln(2)+\ln(1/2)}$	(n) $e^{2\ln(2)-3\ln(3)}$	(o) $\ln\left(e^{\ln(e^2)}\right)$	(p) $e^{-\ln(e^{\ln(3)})}$

2. Find dy/dx for each of the following functions.

(a) $y = \ln(3x)$	(d) $y = \ln(2^x)$	(g) $y = x\ln(x)$	(j) $y = x\ln(x) - x$
(b) $y = 17\ln(x)$	(e) $y = \pi\ln(3e^{4s})$	(h) $y = \cos(\ln(x\sin(x)))$	(k) $y = e^{\sqrt{\ln(x)}}$
(c) $y = \ln(e^{w})$	(f) $y = \ln(4 + 3x^2)$	(i) $y = 1/\ln(1/x)$	(l) $y = \ln(2)$

3. Use properties of e^x, and ideas like those in the "proof of the second property" on page 140, to *prove* the following properties of the logarithm. (Remember that $\ln(b) = a$ means $b = e^a$.) For an ill

(a) $\ln(1) = 0$.

(b) $\ln(m/n) = \ln(m) - \ln(n)$.

(c) $\ln(m^n) = n\ln(m)$.

Part 2: Modeling growth and decay with the exponential function

4. The rate of growth of the population of a particular country is proportional to the population. The last two censuses determined that the population in 1980 was 40,000,000, and in 1985 it was 45,000,000. What will the population be in 1995?

5. Suppose a bacterial population grows so that its mass is

$$P(t) = 200e^{.12t} \quad \text{grams}$$

after t hours. Its initial mass is $P(0) = 200$ grams. When will its mass double, to 400 grams? How much longer will it take to double again, to 800 grams? After the population reaches 800 grams, how long will it take for yet another doubling to happen? What is the *doubling time* of this population?

6. Suppose a beam of X-rays whose intensity is A rads (the "rad" is a unit of radiation) falls perpendicularly on a heavy concrete wall. After the rays have penetrated s feet of the wall, the radiation intensity has fallen to

$$R(s) = Ae^{-.35s} \quad \text{rads.}$$

What is the radiation intensity 3 inches inside the wall; 18 inches? (Your answers will be expressed in terms of A.) How far into the wall must the rays travel before their intensity is cut in half, to $A/2$? How much further before the intensity is $A/4$?

7. Virtually all living things take up carbon as they grow. This carbon comes in two principal forms: normal, stable carbon – C^{12} – and radioactive carbon – C^{14}. C^{14} decays into C^{12} at a rate proportional to the amount of C^{14} remaining. While the organism is alive, this lost C^{14} is continually replenished. After the organism dies, though, the C^{14} is no longer replaced, so the percentage of C^{14} decreases exponentially over time. It is found that after 5730 years, half the original C^{14} remains. If an archaeologist finds a bone with only 20% of the original C^{14} present, how old is it?

8. The human population of the world appears to be growing exponentially. If there were 2.5 billion people in 1960, and 3.5 billion in 1980, how many will there be in 2010?

9. If bacteria increase at a rate proportional to the current number, how long will it take 1000 bacteria to increase to 10,000 if it takes them 17 minutes to increase to 2000?

10. Suppose sugar in water dissolves at a rate proportional to the amount left undissolved. If 40 lb. of sugar reduces to 12 lb. in 4 hours, how long will you have to wait until 99% of the sugar is dissolved?

11. Atmospheric pressure is a function of altitude. Assume that at any given altitude the rate of change of pressure with altitude is proportional to the pressure there. If the barometer reads 30 psi (pounds per square inch) at sea level and 24 psi at 6000 feet above sea level, how high are you when the barometer reads 20 psi?

12. (a) An important concept in many economic analyses is the idea of **present value**. It is used to compare the values of different possible payments made at different times. As a simple example, suppose you had a small wood lot and had the choice of selling the timber on it now for \$5,000 or waiting 10 years for the trees to get larger, at which point you estimate the timber could be sold for \$8,000. To compare these two options, you need to convert the prospect of \$8,000 ten years from now into an equivalent amount of money now – its present value. This is the amount of money you would need to invest now to have \$8,000 in 10 years. Suppose you thought you could invest money at an annual interest rate of 4% compounded continuously. If you invested \$5,000 now at this rate, then in 10 years you would have $5000\,e^{.4}$ = \$7,459.12. That is, \$5,000

now is worth \$7,459.12 in 10 years – both amounts have the same present value. Clearly \$8,000 in 10 years must have a slightly greater present value under the assumption of a 4% annual interest rate. What is it?

(b) On the other hand, if you can get a higher interest rate than 4%, the present value of the \$8,000 will be much less. What is the present value of a payment of \$8,000 ten years from now if the annual interest rate is 8%?

(c) At what interest rate do \$5,000 now and \$8,000 in ten years have the same present value?

Part 3: Numerical computations

13. (a) In the text we noted that the function $\ln x$ is the solution to the initial-value problem

$$\frac{dy}{dx} = \frac{1}{x} \qquad y(1) = 0,$$

so that we can use Euler's method to compute values for $\ln x$. Use this method to evaluate $\ln 2$ to 3 decimal places. What value of Δx gives the desired accuracy?

(b) If you now wanted to calculate $\ln 6$ to 3 decimals, can you think of a better way to do it than simply starting at $x = 1$ and running Euler's method out to $x = 6$? Remember the basic properties of logarithms, and figure out a way to use the results of part (a).

(c) Suppose you had figured out that $\ln 2 = 0.693147\ldots$. How would you use Euler's method to calculate $\ln 1300$ quickly? You might find the fact that $2^{10} = 1024$ helpful.

14. (a) Use a graphing program to find a good numerical approximation to $(\ln x)'$ at $x = 2$. Make a short table, for decreasing interval sizes Δx, of the quantity $\Delta(\ln x)/\Delta x$.

(b) Use a graphing program to find a good numerical approximation to $(e^x)'$ at $x = \ln(2) = 0.6931\ldots$. Make a short table for decreasing interval sizes Δx, of the quantity $\Delta(e^x)/\Delta x$.

(c) What is the relationship between the values you got in parts (a) and (b)?

Differential equations

15. Find a solution (using $\ln x$) to the differential equation

$$f'(x) = 3/x \quad \text{satisfying} \quad f(1) = 2.$$

16. (a) Find a formula using the natural logarithm function giving the solution of $y' = a/x$ with $y(1) = b$.

(b) Solve $P' = 2/t$ with $P(1) = 5$.