Chapter 3

Differential Equations

The rate equations with which we began our study of calculus are called **differential equations** when we identify the rates of change that appear within them as derivatives of functions. Differential equations are essential tools in many area of mathematics and the sciences. In this chapter we explore three of their important uses:

- **Modeling** problems using differential equations;
- **Solving** differential equations, both through numerical techniques like Euler's method and, where possible, through finding formulas which make the equations true;
- **Defining** new functions by differential equations.

We also introduce two important functions – the **natural exponential function** and the **natural logarithm function** – which play central roles in the theory of solving differential equations. Finally, we introduce the operation of **antidifferentiation** as an important tool for solving some special kinds of differential equations.

3.1 The (natural) exponential function

The equation $\frac{dy}{dt} = ky$

Certain differential equations – in fact, some of the very simplest – arise over and over again in an astonishing variety of contexts. The functions they define are among the most important in mathematics.

One of the most basic differential equations is

$$\frac{dy}{dt} = ky \quad \text{(where } k \text{ is a constant).} \tag{3.1.1}$$

It is also one of the most useful, because *it describes a quantity y that is proportional to its rate of change*. Many real-world phenomena evolve in this way, at least roughly. For example: in Section 1.5, we used this differential equation to model the population of Poland, as well as (in Part 3 of the Exercises for that section) bacterial growth and radioactive decay. Later in this chapter we will use it to model other phenomena, such as a rabbit population, how money accrues interest in a bank, and how radiation penetrates solid objects.

We've already encountered certain solutions to the differential equation (3.1.1). Specifically: recall from Section 2.4 that, if $y = b^t$, then

$$\frac{dy}{dt} = \ln(b)b^t, \tag{3.1.2}$$

where

$$\ln(b) = \lim_{\Delta x \to 0} \frac{b^{\Delta x} - 1}{\Delta x}. \tag{3.1.3}$$

But note that the quantity b^t on the right-hand side of (3.1.2) is what we called y in the first place. So (3.1.2) reads

$$\frac{dy}{dt} = \ln(b)y. \tag{3.1.4}$$

In other words, $y = b^t$ *is* a solution to the differential equation (3.1.1), in the case where $k = \ln(b)$.

To study these solutions more systematically, it will be convenient, first, to consider more closely the case $k = 1$.

The equation $\dfrac{dy}{dt} = y$, and the natural exponential function

In Section 2.4 we noted that, by equation (3.1.3) above and the meaning of limits, we have

$$\ln(b) \approx \frac{b^{\Delta x} - 1}{\Delta x} \tag{3.1.5}$$

for Δx small. We used this observation, with $\Delta x = 0.000001$, to find that $\ln(2) \approx 0.69314$. Of course, this number is less than one. A similar approximation shows that $\ln(3) \approx 1.09861$, which is, of course, greater than one. So we might expect that, somewhere between 2 and 3, there is a base b such that $\ln(b)$ exactly *equals* one.

We would be correct! We'll justify this in a moment, but first, taking on faith for the moment that there *is* such a base b, and only one, let's give this base a name: let's call it e.

Definition 3.1.1. The number e is the unique real number satisfying $\ln(e) = 1$.

Some important properties of the number e are as follows.

1. Since $d[b^t]/dt = \ln(b)b^t$ for any positive base b (as noted above), and since $\ln(e) = 1$, we have

$$\boldsymbol{\frac{d}{dt}[e^t] = \ln(e)e^t = 1 \cdot e^t = e^t.}$$

The derivative of e^t is e^t

In other words, **the natural exponential function $y = e^t$ is equal to its own derivative**. This property of the function $y = e^t$ is largely what makes the base e so useful and ubiquitous. (The *raison d'etre* of the base e is a calculus thing!)

In fact, the natural exponential function is *so* natural that we typically call it, simply, **the exponential function** (to the chagrin, perhaps, of $y = b^t$ for other values of b).

Further, as we'll soon see, solutions to the differential equation (3.1.1) may be written explicitly and simply in terms of t, k, and e.

2. Putting $b = e$ into equation (3.1.5) above, and recalling that $\ln(e) = 1$, we get

$$1 \approx \frac{e^{\Delta x} - 1}{\Delta x} \tag{3.1.6}$$

for Δx small. We use this equation to approximate e, as follows. We multiply both sides of (3.1.6) by Δx, add 1 to both sides of our result, and then raise both sides to the power of $1/\Delta x$. We find that

$$e \approx (1 + \Delta x)^{1/\Delta x} \tag{3.1.7}$$

for Δx small.

For example, using $\Delta x = 0.000001$ gives

$$e \approx (1.000001)^{1{,}000{,}000} = 2.71828...$$

And we could get a better approximation with a smaller Δx. (It turns out that e is irrational, so its decimal expansion has infinitely many places.)

The fact (which we won't prove) that the right-hand side of (3.1.7) *has* a limit as $\Delta x \to 0$ is what tells us that the base e exists, and it's not hard to show, using essentially the above arguments, that this base e is the only one whose natural logarithm equals one.

The use of the symbol e to denote the base described above dates back to a paper that Euler wrote at age 21, entitled *Meditatio in experimenta explosione tormentorum nuper instituta* (Meditation upon recent experiments on the firing of cannons), where the symbol e was used sixteen times. It is now in universal use. The number e is, like π, one of the most important and ubiquitous in mathematics.

Here are some examples involving differentiation of the (natural) exponential function.

Example 3.1.1. (a) Find: (i) $\frac{d}{dt}\left[e^{4t}\right]$; (ii) $\frac{d}{dz}\left[e^{3\sin(z)}\right]$; (iii) $\frac{d}{dz}\left[3\sin(e^z)\right]$.

(b) Write down the microscope equation for $f(x) = e^x$ at $x = 0$, and use this result to approximate $e^{0.03}$.

Solution. (a)(i) The derivative of e^t, with respect to t, is e^t, so by the chain rule, the derivative of e to a function of t, with respect to t, is e to that same function of t, times the derivative of that function of t. So

$$\frac{d}{dt}\left[e^{4t}\right] = e^{4t}\frac{d}{dt}[4t] = e^{4t} \cdot 4 = 4e^{4t}.$$

(ii) By similar reasoning,

$$\frac{d}{dz}\left[e^{3\sin(z)}\right] = e^{3\sin(z)}\frac{d}{dz}\left[3\sin(z)\right] = e^{3\sin(z)} \cdot 3\cos(z) = 3\cos(z)e^{3\sin(z)}.$$

(iii)The derivative of the sine function is the cosine function, so by the chain rule,

$$\frac{d}{dz}\left[3\sin(e^z)\right] = 3\frac{d}{dz}\left[\sin(e^z)\right] = 3\cos(e^z)\frac{d}{dz}\left[e^z\right] = 3\cos(e^z)e^z.$$

(b) Let $f(x) = e^x$. Then $f'(x) = e^x$. If $a = 0$, then $f(a) = e^0 = 1$ and $f'(0) = e^0 = 1$. So the microscope equation reads

$$\begin{aligned} f(x) &\approx f(a) + f'(a)\Delta x \\ e^{\Delta x} &\approx 1 + \Delta x. \end{aligned}$$

So

$$e^{0.03} \approx 1.03.$$

(A calculator gives $e^{0.03} = 1.0304....$)

The equation $\dfrac{dy}{dt} = ky$, again

In Example 3.1.1 above, we differentiated e to a constant times the independent variable. Notice that the result of this differentiation was that constant times the function we started with!

More generally, consider the function

$$y = Ce^{kt},$$

where both C and k are constants, and $k > 0$. We claim that y has two important features: (i) the rate of growth of y is proportional to y, with constant of proportionality k; and (ii) initially, meaning when $t = 0$, y is equal to the contant C.

We prove these claims as follows: first, by the chain rule,

$$\frac{dy}{dt} = \frac{d}{dt}\left[Ce^{kt}\right] = C\frac{d}{dt}\left[e^{kt}\right] = Ce^{kt}\frac{d}{dt}\left[kt\right] = Ce^{kt} \cdot k = y \cdot k = ky.$$

And second,

$$y(0) = Ce^{k \cdot 0} = Ce^0 = C \cdot 1 = C$$

(since any positive number to the 0th power is one).

We summarize the above in the following.

$y = Ce^{kt}$ is the solution to the initial value problem

$$\frac{dy}{dt} = ky; \qquad y(0) = C.$$

Here k and C are constants, and $k > 0$.

Solution to the "exponential growth" initial value problem

A few remarks are in order.

(a) Recall that, by "initial value problem," we mean one or more differential equations, together with one or more *initial conditions*, meaning specification of how things look at some particular point in time (often at $t = 0$). In the exponential growth initial value problem, the equation $dy/dt = ky$ is of course the differential equation, and $y(0) = C$ the initial condition.

(b) We have shown that $y = Ce^{kt}$ is **a** solution to the exponential growth initial value problem, but here, we are saying more. We are saying it is **the** solution, meaning there are no others. We have not proved this, but it follows from general results in the theory of differential equations, and we will take it on faith.

Example 3.1.2. A population P grows at a yearly rate equal to 0.3 times the population size. Assume that $P = 100{,}000$ when $t = 0$.

(a) Write down the initial value problem for the population P.

(b) Write down a formula for $P(t)$.

(c) What is the population after four years?

Solution. (a) We have

$$\frac{dP}{dt} = 0.3P; \qquad P(0) = 100{,}000.$$

(b) By the above-stated results concerning the solution to the exponential growth initial value problem,

$$P(t) = 100{,}000e^{0.3t}.$$

Here, t is in years and P is in individuals.

(c) $P(4) = 100{,}000e^{0.3\times 4} = 332{,}012$ individuals (to the nearest whole number of individuals).

In part (c) of the above example, we found the value of $P(t)$ corresponding to a particular value of t. In a later section, we'll see how to do an "inverse" process: we'll see how to find the value of t that corresponds to a particular value of $P(t)$. That is, we'll answer a "how long" question, as opposed to a "how large" question. (We'll solve for the independent variable instead of the dependent variable.)

We now note that, strictly speaking, $y = Ce^{kt}$ satisfies the initial value problem given by $dy/dt = ky$ and $y(0) = C$ for *any* real number k, not just when k is positive. However, if k is negative, then the equation $dy/dt = ky$ tells us that y has a negative derivative (assuming y itself is positive), so that y is decreasing. In this case, we have *decay* rather than growth, and it's a bit jarring to still refer to the "exponential growth" initial value problem. So we typically formulate decay situations a bit differently. Namely, we still require that k be positive, but we write $-k$ instead of k for our constant of proportionality. Our solution, then, also entails a factor of $-k$ instead of k, and we have the following.

$\boldsymbol{y = Ce^{-kt}}$ **is the solution to the initial value problem** $$\frac{dy}{dt} = -ky; \qquad y(0) = C.$$ **Here k and C are constants, and $k > 0$.**

Solution to the "exponential decay" initial value problem

Example 3.1.3. Sugar dissolves in water at a rate proportional to the amount present. Write an equation for the amount $S(t)$ of sugar remaining after t minutes, in terms of a " per unit decay rate" k and an initial amount S_0. Assuming that $S(t)$ is measured in pounds, what are the units for k?

Solution. We're given that $S'(t) = -kS(t)$ (the minus sign indicates decay), and that $S(0) = S_0$. So, by the above-stated results concerning the exponential decay initial value problem,

$$S(t) = S_0 e^{-kt}.$$

The units for k are given by the original differential equation: if $S'(t) = -kS$ and $S(t)$ is measured in pounds, and t is in minutes, then k must be in minutes^{-1}, so that units match up on the two sides of this differential equation.

In a later section, we'll see how to find a numerical value for the parameter k, given certain additional information (such as, for example, the amount of salt remaining five minutes later).

Basic properties of the (natural) exponential function

The function $y = e^x$ has a number of useful properties. Among these are the following.

(i) $e^0 = 1$.

(ii) $e^x > 0$ for all real numbers x.

(iii) $e^{-x} = 1/e^x$ for all real numbers x.

(iv) $e^{x+y} = e^x e^y$ for all real numbers x and y.

(v) $e^{x-y} = e^x/e^y$ for all real numbers x and y.

(vi) $(e^r)^s = e^{rs}$ for all real numbers r and s.

(vii) $\dfrac{d}{dx}[e^x] = e^x$.

(viii) The graph of $y = e^x$ looks like this:

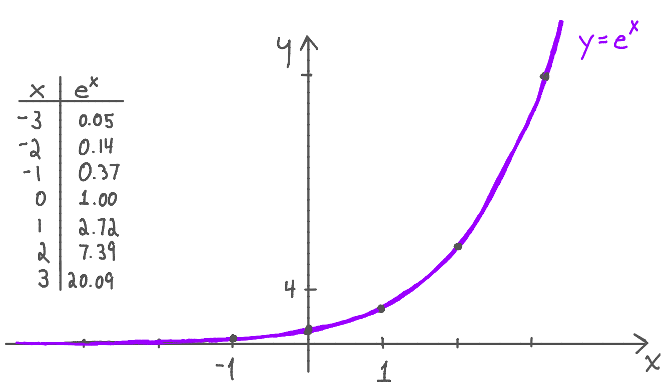

Figure 3.1. The natural exponential function $y = e^x$

(ix) $\lim_{x \to +\infty} e^x = +\infty$. (x) $\lim_{x \to -\infty} e^x = 0$.

The first six of these properties are still true if "e" is replaced by "b," where b is any positive number. The seventh and eighth properties are, strictly speaking, unique to the base e, though they hold for other bases if suitably modified. (Food for thought: what modifications are required?)

The ninth property tells us that e^x increases without bound, or "blows up" (in the positive, rather than the negative, vertical direction), as we go further out along the positive x axis. The tenth property tells us that e^x gets arbitrarily close to zero as we go further out along the *negative* x axis.

Properties (ix) and (x) remain true with e replaced by b, **as long as b is larger than 1**. If $b \le 1$, then properties (ix) and (x) require modification. See the exercises below.

Example 3.1.4. Find $\dfrac{d}{dx}\left[e^x e^{7-x}\right]$ in two ways:

(a) Differentiate and then simplify;

(b) Simplify and then differentiate.

Solution. (a) By the product and chain rules,

$$\begin{aligned}\frac{d}{dx}\left[e^x e^{7-x}\right] &= e^x \frac{d}{dx}\left[e^{7-x}\right] + e^{7-x}\frac{d}{dx}\left[e^x\right] = e^x e^{7-x}\frac{d}{dx}\left[7-x\right] + e^{7-x}e^x \\ &= e^x e^{7-x}(-1) + e^{7-x}e^x = 0,\end{aligned}$$

since the two summands cancel.

(b) By the above property (iv), we have $e^x e^{7-x} = e^{x+7-x} = e^7$, which is just a *constant.* So

$$\frac{d}{dx}\left[e^x e^{7-x}\right] = \frac{d}{dx}\left[e^7\right] = 0$$

(again).

Exponential growth

The function $\exp(x) = e^x$, like polynomials and the sine and cosine functions, is defined for all real numbers. Nevertheless, it behaves in a way that is quite different from any of those functions.

One difference occurs when x is large, either positive or negative. The sine function and the cosine function stay bounded between $+1$ and -1 over their entire domain. By contrast, every polynomial "blows up" as $x \to \pm\infty$. In this regard, the exponential function is a hybrid. As $x \to -\infty$, $\exp(x) \to 0$. As $x \to +\infty$, however, $\exp(x) \to +\infty$.

Let's look more closely at what happens to power functions x^n and the exponential function e^x as $x \to \infty$. Both kinds of functions "blow up" but they do so at quite different rates, as we shall see. Before we compare power and exponential functions directly, let's compare one power of x with another – say x^2 with x^5. As $x \to \infty$, both x^2 and x^5 get very large. However, x^2 is only a small fraction of the size of x^5, and that fraction gets smaller, the larger x is. The following table demonstrates this. Even though x^2 becomes enormous, we interpret the fact that $x^2/x^5 \to 0$ to mean that x^2 *grows more slowly than* x^5.

x	x^2	x^5	x^2/x^5
10	10^2	10^5	10^{-3}
100	10^4	10^{10}	10^{-6}
1000	10^6	10^{15}	10^{-9}
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
∞	∞	∞	0

It should be clear to you that we can compare *any* two powers of x this way. We will find that x^p grows more slowly than x^q if, and only if, $p < q$. To prove this, we must see what happens to the ratio x^p/x^q, as $x \to +\infty$. We can write $x^p/x^q = 1/x^{q-p}$, and the exponent $q-p$ that appears here is positive, because $q > p$. Consequently, as $x \to \infty$, $x^{q-p} \to \infty$ as well, and therefore $1/x^{q-p} \to 0$. This completes the proof.

How does e^x compare to x^p? To make it tough on e^x, let's compare it to x^{50}. We know already that x^{50} grows faster than any lower power of x. The table below compares x^{50} to e^x. However, the numbers involved are so large that the table shows only their *order of magnitude* – that is, the number of digits they contain. At the start, x^{50} is *much* larger than e^x. However, by the time $x = 500$, the ratio x^{50}/e^x is so small its first 82 decimal places are zero!

x	x^{50}	e^x	x^{50}/e^x
100	$\sim 10^{100}$	$\sim 10^{43}$	$\sim 10^{56}$
200	10^{115}	10^{86}	10^{28}
300	10^{123}	10^{130}	10^{-7}
400	10^{130}	10^{173}	10^{-44}
500	10^{134}	10^{217}	10^{-83}
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
∞	∞	∞	0

So x^{50} grows more slowly than e^x, and so does any lower power of x. Perhaps a higher power of x would do better. It does, but ultimately the ratio $x^p/e^x \to 0$, no matter how large the power p is. We don't yet have all the tools needed to prove this, but we will after we introduce the logarithm function in the next section.

The speed of exponential growth has had an impact in computer science. In many cases, the number of operations needed to calculate a particular quantity is a power of the number of digits of precision required in the answer. Sometimes, though, the number of operations is an *exponential* function of the number of digits. When that happens, the number of operations can quickly exceed the capacity of the computer. In this way, some problems that can be solved by an algorithm that is straightforward in theoretical terms are intractable in practical terms.

Exercises

Part 1: The functions b^x

1. What are the analogs of properties (ix) and (x) of the exponential function above, if we replace e with the base $b = 1$? That is, what are

$$\lim_{x\to+\infty} 1^x \quad \text{and} \quad \lim_{x\to-\infty} 1^x?$$

2. (a) Use a graphing utility to plot $y = 3^x$ and $y = (1/3)^x$ on the same axes. Describe how these two functions compare, graphically.

(b) Repeat part (a) for $y = 7^x$ and $y = (1/7)^x$.

(c) We noted above that properties (ix) and (x) of the function $y = e^x$ still hold if we replace e by b, as long as $b > 1$. In light of parts (a) and (b) of this exercise, what do you think the analogous properties are for $b < 1$? Hint: recall that $b^{-x} = 1/b^x$.

Part 2: Differentiating exponential functions

3. Differentiate the following functions.

(a) $f(x) = 7e^{3x}$

(b) $y = Ce^{kx}$, where C and k are constants.

(c) $g(t) = 1.5e^t$

(d) $q = 1.5e^{2t}$

(e) $r(x) = 2e^{3x} - 3e^{2x}$

(f) $z(t) = e^{\cos t}$

(g) $y = x^4e^{4x}$

(h) $f(v) = \dfrac{e^v}{e^v + 1}$

(i) $y = \tan(x^5e^{5x})$

(j) $y = e^{e^x}$

(k) $q = (e^\pi)^\pi$

Part 3: Powers of e

4. Rewrite each of the following quantities as a whole number, or as e to a single expression. For each, please state which of the above properties (i)–(vi) of the exponential function you have used. The first one has been done for you, to illustrate what is meant.

(a) $(e^x)^2/e^y$ (**Solution:** $(e^x)^2/e^y = e^{2x}/e^y = e^{2x-y}$, by properties (vi) and (v).)

(b) $\dfrac{e^3e^2}{e^5}$

(c) $e^{x^2}/(e^x)^2$ (Note: e^{x^2} means $e^{(x^2)}$, not $(e^x)^2$.)

(d) $\big((e^2e^3e^4)/e\big)/e^3$

(e) $\left(e^{y^2-5y}\right)^{1/y}e^5/e^y$

(f) $\left((e^3)^y(e^{-y})^3/e^x\right)^2$

Part 4: Solving $y' = ky$ using e^t

5. **Poland**. Refer to Example 1.5.3 in Section 1.5 above.

(a) Write out the initial value problem for the population P given in this example.

(b) Write a formula for the solution of this initial value problem.

(c) Use your formula from part (b) (and a calculator) to find the population of Poland in the year 2005. What was the population in 1965?

6. **Bacterial growth**. Refer to Exercise 13 of Section 1.5.

(a) Assuming that we begin with the colony of bacteria weighing 32 grams, write out the initial value problem that summarizes the information about the weight P of the colony.

(b) Write a formula for the solution P of this initial value problem.

(c) How much does the colony weigh after 30 minutes? after 2 hours?

7. **Radioactivity**. Refer to Exercise 14 of Section 1.5.

(a) Assuming that, when we begin, the sample of radium weighs 1 gram, write out the initial value problem that summarizes the information about the weight R of the sample.

(b) How much did the sample weigh 20 years ago? How much will it weigh 200 years hence?

8. **Intensity of radiation**. As gamma rays travel through an object, their intensity I decreases with the distance x that they have travelled. This is called **absorption**. The absorption rate

dI/dx is proportional to the intensity. For some materials the multiplier in this proportion is large; they are used as radiation shields.

(a) Write down a differential equation which models the intensity of gamma rays $I(x)$ as a function of distance x.

(b) Some materials, such as lead, are better shields than others, such as air. How would this difference be expressed in your differential equation?

(c) Assume the unshielded intensity of the gamma rays is I_0. Write a formula for the intensity I in terms of the distance x and verify that it gives a solution of the initial value problem.

9. In this problem you will find a solution for the initial value problem $y' = ky$ and $y(t_0) = C$. (Notice that this isn't the original initial value problem, because t_0 was 0 originally.)

(a) Explain why you may assume $y = Ae^{kt}$ for some constant A.

(b) Find A in terms of k, C and t_0.

Part 5: Solving other differential equations

10. (a) **Newton's law of cooling**. Verify that

$$Q(t) = 70e^{-0.1\,t} + 20$$

is a solution to the initial value problem $Q'(t) = -0.1(Q-20)$, $Q(0) = 90$. What is the relationship between this formula and the one found in problem 11 in section 2?

(b) Verify that

$$Q(t) = (Q_0 - A)e^{-kt} + A$$

is a solution to the the initial value problem $Q'(t) = -k(Q-A)$, $Q(0) = B$. What is the relationship between this formula and the one found in problem 13 in section 2?

11. In *An Essay on the Principle of Population*, written in 1798, the British economist Thomas Robert Malthus (1766–1834) argued that food supplies grow at a constant rate, while human populations naturally grow at a constant *per capita* rate. He therefore predicted that human populations would inevitably run out of food (unless population growth was suppressed by unnatural means).

(a) Write differential equations for the size P of a human population and the size F of the food supply that reflect Malthus' assumptions about growth rates.

(b) Keep track of the population in millions, and measure the food supply in millions of units, where one unit of food feeds one person for one year. Malthus' data suggested to him that the food supply in Great Britain was growing at about .28 million units per year and the per capita growth rate of the population was 2.8% per year. Let $t = 0$ be the year 1798, when Malthus estimated the population of the British Isles was $P = 7$ million people. He assumed his countrymen were on average adequately nourished, so he estimated that the food supply was $F = 7$ million units of food. Using these values, write formulas for the solutions $P = P(t)$ and $F = F(t)$ of the differential equations in (a).

(c) Use the formulas in (b) to calculate the amount of food and the population at 25 year intervals for 100 years. Use these values to help you sketch graphs of $P = P(t)$ and $F = F(t)$ on the same axes.

(d) The per capita food supply in any year equals the ratio $F(t)/P(t)$. What happens to this ratio as t grows larger and larger? (Use your graphs in (c) to assist your explanation.) Do your results support Malthus's prediction? Explain.

Part 6: Interest rates

Bank advertisements sometimes look like this:

Civic Bank and Trust

- Annual rate of interest 6%.
- Compounded monthly.
- Effective rate of interest 6.17%.

The first item seems very straightforward. The bank pays 6% interest per year. Thus if you deposit \$100.00 for one year then at the end of the year you would expect to have \$106.00. Mathematically this is the simplest way to compute interest; each year add 6% to the account. The biggest problem with this is that people often make deposits for odd fractions of a year, so if interest were paid only once each year then a depositor who withdrew her money after 11 months would receive no interest. To avoid this problem banks usually compute and pay interest more frequently. The Civic Bank and Trust advertises interest **compounded** monthly. This means that the bank computes interest each month and credits it (that is, adds it) to the account.

Month	Start	Interest	End
1	\$100.0000	.5000	\$100.5000
2	\$100.5000	.5025	\$101.0025
3	\$101.0025	.5050	\$101.5075
4	\$101.5075	.5075	\$102.0151
5	\$102.0151	.5101	\$102.5251
6	\$102.5251	.5126	\$103.0378
7	\$103.0378	.5152	\$103.5529
8	\$103.5529	.5178	\$104.0707
9	\$104.0707	.5204	\$104.5911
10	\$104.5911	.5230	\$105.1140
11	\$105.1140	.5256	\$105.6396
12	\$105.6396	.5282	\$106.1678

Since this particular account pays interest at the rate of 6% per year and there are 12 months in a year the interest rate is 6%/12 = 0.5% per month. The following table shows the interest computations for one year for a bank account earning 6% annual interest compounded monthly.

Notice that at the end of the year the account contains $106.17. It has effectively earned 6.17% interest. This is the meaning of the advertised *effective* rate of interest. The reason that the effective rate of interest is higher than the original rate of interest is that the interest earned each month itself earns interest in each succeeding month. (We first encountered this phenomenon when we were trying to follow the values of S, I, and R into the future.) The difference between the original rate of interest and the effective rate can be very significant. Banks routinely advertise the effective rate to attract depositors. Of course, banks do the same computations for loans. They rarely advertise the effective rate of interest for loans because customers might be repelled by the true cost of borrowing.

The effective rate of interest can be computed much more quickly than we did in the previous table. Let R denote the annual interest rate as a decimal. For example, if the interest rate is 6% then $R = 0.06$. If interest is compounded n times per year then each time it is compounded the interest rate is R/n. Thus each time you compound the interest you compute

$$V + \left(\frac{R}{n}\right) V = \left(1 + \frac{R}{n}\right) V$$

where V is the value of the current deposit. This computation is done n times during the course of a year. So, if the original deposit has value V, after one year it will be worth

$$\left(1 + \frac{R}{n}\right)^n V.$$

For our example above this works out to

$$\left(1 + \frac{0.06}{12}\right)^{12} V = 1.061678\, V$$

and the effective interest rate is 6.1678%.

Many banks now compound interest daily. Some even compound interest *continuously*. The value of a deposit in an account with interest compounded continuously at the rate of 6% per year, for example, grows according to the differential equation

$$V' = 0.06V.$$

12. Many credit cards charge interest at an annual rate of 18%. If this rate were compounded monthly what would the effective annual rate be?

13. In fact many credit cards compound interest daily. What is the effective rate of interest for 18% interest compounded daily? Assume that there are 365 days in a year.

14. The assumption that a year has 365 days is, in fact, *not* made by banks. They figure every one of the 12 months has 30 days, so their year is 360 days long. This practice stem from the time when interest computations were done by hand or by tables, so simplicity won out over precision. Therefore when banks compute interest they find the daily rate of interest by dividing the annual rate of interest by 360. For example, if the annual rate of interest is 18% then the daily rate of interest is 0.05%. Find the effective rate of interest for 18% compounded 360 times per year.

15. In fact, once they've obtained the daily rate as 1/360-th of the annual rate, banks then compute the interest *every* day of the year. They compound the interest 365 times. Find the effective rate of interest if the annual rate of interest is 18% and the computations are done by banks. First, compute the daily rate by dividing the annual rate by 360 and then compute interest using this daily rate 365 times.

16. Consider the following advertisement.

Civic Bank and Trust

- Annual rate of interest 6%.
- Compounded daily.
- Effective rate of interest 6.2716%.

Find the effective rate of interest for an annual rate of 6% compounded daily in the straightforward way – using 1/365-th of the annual rate 365 times. Then do the computations the way they are done in a bank. Compare your two answers.

17. There are two advertisements in the newspaper for savings accounts in two different banks. The first offers 6% interest compounded quarterly (that is, four times per year). The second offers 5.5% interest compounded continuously. Which account is better? Explain.